Integration Guide

Heidelberg Integration Bee 2026

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April 16, 2026

Introduction

This guide is designed to provide an overview of the basic techniques of integration. Although not every technique needs to be remembered extensively, it is good to have in mind what options there are to approach different integrals. To give an impression of the different methods, each integration technique will be applied to an example and additional integrals will be given to practise.

It also serves as a support for the Heidelberg Integration Bee 2026.

Techniques

1 Partial Integration

$$\int_a^b f'(x)g(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f(x)g'(x)\,dx \tag{1.1}$$

$$\iff \int u\,dv = uv - \int v\,du \tag{1.2}$$

Proof. Follows from product rule:

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x) \tag{1.3}$$

Integrating both sides:

$$\int_a^b \frac{d}{dx}(f(x)g(x))\,dx = f(x)g(x)\Big|_a^b = \int_a^b f'(x)g(x)\,dx + \int_a^b f(x)g'(x)\,dx \tag{1.4}$$

$$\implies \int_a^b f'(x)g(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f(x)g'(x)\,dx \tag{1.5}$$

□

Example 1.1

$$I = \int_0^{\frac{\pi}{2}} \cos(x)\sin(x)\,dx \tag{1.6}$$

For the functions in the integrand, we choose:

$$f'(x) = \cos(x) \qquad g(x) = \sin(x) \tag{1.7}$$

The derivate and an antiderivative of these functions are:

$$f(x) = \sin(x) \qquad g'(x) = \cos(x) \tag{1.8}$$

Now using partial integration, our integral becomes:

$$\begin{aligned} I = \int_0^{\frac{\pi}{2}} \cos(x)\sin(x)\,dx &= \sin(x)\cdot\sin(x)\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin(x)\cdot\cos(x)\,dx \\ &= \sin^2(x)\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \cos(x)\sin(x)\,dx \\ &= \sin^2(x)\Big|_0^{\frac{\pi}{2}} - I \end{aligned} \tag{1.9}$$

Here we notice that the new integral is the negative of our original, so we can add it on both sides, resulting in

$$2I = 2\int_0^{\frac{\pi}{2}} \cos(x)\sin(x)\,dx = \sin^2(x)\Big|_0^{\frac{\pi}{2}} \tag{1.10}$$

dividing by two gives us

$$I = \int_0^{\frac{\pi}{2}} \cos(x)\sin(x)\,dx = \frac{1}{2}\sin^2(x)\Big|_0^{\frac{\pi}{2}} = \frac{1}{2} \tag{1.11}$$

Example 1.2

$$\int x \arctan x \, dx \tag{1.12}$$

Choosing:

$$u = \arctan x \qquad\qquad dv = x\, dx \tag{1.13}$$

We get for our derivative:

$$du = \frac{1}{x^2+1}\, dx \tag{1.14}$$

For v we need an antiderivative of x. A possible choice is $\frac{1}{2}x^2$. This way our integral would look like:

$$\int x \arctan x \, dx = \frac{1}{2}x^2 \arctan x - \int \frac{\frac{1}{2}x^2}{x^2+1}\, dx \tag{1.15}$$

But now we would have to integrate the fraction of two different polynomials. Instead we could also choose a different antiderivative of x:

$$v = \frac{x^2+1}{2} \tag{1.16}$$

Now our Integral becomes much cleaner:

$$\begin{aligned}\int x \arctan x \, dx &= \frac{x^2+1}{2} \arctan x - \int \frac{x^2+1}{2} \frac{1}{x^2+1}\, dx \\ &= \frac{1}{2}(x^2+1) \arctan x - \frac{1}{2}\int dx \\ &= \frac{1}{2}((x^2+1)\arctan x - x) + \text{constant}\end{aligned} \tag{1.17}$$

Practice Integrals:

a) $\displaystyle\int_1^2 \frac{\ln(x)}{x^2}\, dx$

b) $\displaystyle\int \cos(x)\ln(\sin(x))\, dx$

c) $\displaystyle\int_0^\pi \sin^2(x)\, dx$

d) $\displaystyle\int \ln(x)\, dx$

e) $\displaystyle\int \sin(x)e^x\, dx$

f) $\displaystyle\int_0^\infty t^n\, e^{-t}\, dt$ for $n \in \mathbb{N}$

2 Substitution

$$\int_a^b f(\varphi(x))\varphi'(x)\, dx = \int_{\varphi(a)}^{\varphi(b)} f(u)\, du \tag{2.1}$$

Proof. Follows from chain rule: Let F be the antiderivative of f

$$\frac{d}{dx}(F \circ \varphi)(x) = \frac{d}{dx}F(\varphi(x)) = F'(\varphi(x))\varphi'(x) = f(\varphi(x))\varphi'(x) \tag{2.2}$$

Finally, integrating both sides and applying the fundamental theorem of calculus:

$$\begin{aligned}\int_a^b f(\varphi(x))\varphi'(x)\,dx &= \int_a^b \frac{d}{dx}(F\circ\varphi)(x)\,dx = (F\circ\varphi)(x)\Big|_a^b \\ &= F(\varphi(b)) - F(\varphi(a)) \quad = F\Big|_{\varphi(a)}^{\varphi(b)} \\ &= \int_{\varphi(a)}^{\varphi(b)} F'(u)\,du \qquad = \int_{\varphi(a)}^{\varphi(b)} f(u)\,du\end{aligned} \tag{2.3}$$

□

Example 2.1

$$\int_a^b \frac{\ln x}{x}\,dx \tag{2.4}$$

In the integral you can see both, $\ln x$ and its derivative $\frac{1}{x}$. We therefore choose

$$u = \ln x \tag{2.5}$$

with the differential

$$\frac{du}{dx} = \frac{1}{x} \quad \Longrightarrow \quad du = \frac{dx}{x} \tag{2.6}$$

The integral then becomes:

$$\int_a^b \ln x\left(\frac{dx}{x}\right) = \int_{\ln a}^{\ln b} u\,du = \frac{1}{2}u^2\Big|_{\ln a}^{\ln b} = \frac{1}{2}(\ln x)^2\Big|_a^b \tag{2.7}$$

Practice Integrals:

a) $\int \sqrt{4-\sqrt{x}}\,dx$

b) $\int \frac{x^2}{\sqrt{1-x}}\,dx$

c) $\int_0^\pi \sin^3 x\,dx$

d) $\int \frac{1}{e^x+e^{-x}}\,dx$

e) $\int \frac{1}{\cos^4(x)}\,dx$

3 King's Rule

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx \tag{3.1}$$

Proof. Follows from substitution with $x = a+b-u$:

$$\begin{aligned}\int_a^b f(x)\,dx &= -\int_{a+b-a}^{a+b-b} f(a+b-u)\,du \\ &= -\int_b^a \quad f(a+b-u)\,du = \int_a^b f(a+b-u)\,du\end{aligned} \tag{3.2}$$

□

Example 3.1

$$\int_0^{2\pi} \frac{\sin^2 x}{x-\pi}\,dx \tag{3.3}$$

Applying the King's Rule we get

$$\int_0^{2\pi} \frac{\sin^2(2\pi - x)}{(2\pi - x) - \pi}\,dx = \int_0^{2\pi} \frac{\sin^2(-x)}{\pi - x}\,dx = -\int_0^{2\pi} \frac{\sin^2 x}{x-\pi}\,dx \tag{3.4}$$

It follows that the whole integral evaluates to 0 because it is its own negative.

Practice Integrals:

a) $\displaystyle\int_2^3 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{5-x}}\,dx$

b) $\displaystyle\int_{-3}^3 \frac{x^2}{1+a^x}\,dx$ for $a > 0$

c) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^n(x)}{\sin^n(x)+\cos^n(x)}\,dx$ for $n \in \mathbb{N}$

d) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+tan^{\pi}(x)}\,dx$

4 Even/Odd Functions

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\displaystyle\int_0^a f(x)\,dx, & \text{if } f \text{ is even} \\ 0, & \text{if } f \text{ is odd} \end{cases} \tag{4.1}$$

$$\iff \quad F(x) = \int_0^x f(y)\,dy \text{ is } \begin{cases} \text{odd}, & \text{if } f \text{ is even} \\ \text{even}, & \text{if } f \text{ is odd} \end{cases} \tag{4.2}$$

Proof. First equation (4.2):
By substituting $y = -u$, $dy = -du$ we get

$$F(x) = \int_0^x f(y)\,dy = -\int_0^{-x} f(-u)\,du = \begin{cases} -\displaystyle\int_0^{-x} f(u)\,du = -F(-x), & \text{if } f \text{ is even} \\ \displaystyle\int_0^{-x} f(u)\,du = F(-x), & \text{if } f \text{ is odd} \end{cases} \tag{4.3}$$

Equation (4.1):
Let F be the antiderivative of f according to equation (4.2):

$$\int_{-a}^{a} f(x)\,dx = F(a) - F(-a) = \begin{cases} F(a) + F(a) = 2\displaystyle\int_0^a f(x)\,dx & \text{if } f \text{ is even} \\ F(a) - F(a) = 0 & \text{if } f \text{ is odd} \end{cases} \tag{4.4}$$

□

Practice Integrals:

a) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^4(x)}{x^3}\,dx$

5 Trigonometric/Hyperbolic Derivatives/Antiderivatives

$\frac{d}{dx} f(x)$	$f(x)$	$\int f(x)\, dx$
$\cos x$	$\sin x$	$-\cos x$
$-\sin x$	$\cos x$	$\sin x$
$1+\tan^2 x = \sec^2 x$	$\tan x$	$-\ln\lvert\cos x\rvert$
$-1-\cot^2 x = -\csc^2 x$	$\cot x$	$\ln\lvert\sin x\rvert$
$-\csc x \cot x$	$\csc x$	$-\ln\lvert\csc x + \cot x\rvert$
$\sec x \tan x$	$\sec x$	$\ln\lvert\sec x + \tan x\rvert$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x$	$\sqrt{1-x^2} + x \arcsin x$
$-\frac{1}{\sqrt{1-x^2}}$	$\arccos x$	$-\sqrt{1-x^2} + x \arccos x$
$\frac{1}{1+x^2}$	$\arctan x$	$-\frac{1}{2}\ln\left(1+x^2\right) + x \arctan x$
$-\frac{1}{1+x^2}$	$\operatorname{arccot} x$	$\frac{1}{2}\ln\left(1+x^2\right) + x \operatorname{arccot} x$

$\frac{d}{dx} f(x)$	$f(x)$	$\int f(x)\, dx$
$\cosh x$	$\sinh x$	$\cosh x$
$\sinh x$	$\cosh x$	$\sinh x$
$1-\tanh^2 x = \frac{1}{\cosh^2 x}$	$\tanh x$	$\ln(\cosh x)$
$-1+\coth^2 x = -\frac{1}{\sinh^2 x}$	$\coth x$	$\ln\lvert\sinh x\rvert$
$\frac{1}{\sqrt{x^2+1}}$	$\operatorname{arsinh} x$	$-\sqrt{x^2+1} + x \operatorname{arsinh} x$
$\frac{1}{\sqrt{x^2-1}}$	$\operatorname{arcosh} x$	$-\sqrt{x^2-1} + x \operatorname{arcosh} x$
$\frac{1}{1-x^2}$	$\operatorname{artanh} x$	$\frac{1}{2}\ln\left(1-x^2\right) + x \operatorname{artanh} x$
$\frac{1}{1-x^2}$	$\operatorname{arcoth} x$	$\frac{1}{2}\ln\lvert 1-x^2\rvert + x \operatorname{arcoth} x$

Practice Integrals:

a) $\int \frac{1}{ax^2+bx+c}\, dx$

b) $\int \sqrt{x^2-1}$

6 Weierstrass Substitution

$$\int_a^b f(\sin x, \cos x)\, dx = \int_{\tan\frac{a}{2}}^{\tan\frac{b}{2}} f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right) \frac{2}{1+t^2}\, dt \qquad \text{for } \lvert a\rvert, \lvert b\rvert \le \pi \tag{6.1}$$

Proof. By Substitution with $t = \tan\frac{x}{2}$, the trigonometric functions become:

$$\sin x = 2\sin\tfrac{x}{2}\cos\tfrac{x}{2} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}} = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2t}{1+t^2} \tag{6.2}$$

$$\cos x = \cos^2\tfrac{x}{2} - \sin^2\tfrac{x}{2} = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}} = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-t^2}{1+t^2} \tag{6.3}$$

And the differential is:

$$\frac{dt}{dx} = \frac{d}{dx}\tan\tfrac{x}{2} = \frac{1+\tan^2\frac{x}{2}}{2} = \frac{1+t^2}{2} \quad\Longrightarrow\quad dx = \frac{2}{1+t^2}dt \tag{6.4}$$

□

Example 6.1

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x+\cos x}\,dx \tag{6.5}$$

Using the formula for the Weierstrass substitution, the integral becomes

$$\begin{aligned}\int_{\tan 0}^{\tan\frac{x}{4}} \frac{1}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,dx &= \int_0^1 \frac{2}{1+t^2+2t+1-t^2}\,dt \\ &= \int_0^1 \frac{1}{1+t}\,dt = \ln(1+t)\Big|_0^1 = \ln 2\end{aligned} \tag{6.6}$$

Practice Integrals:

a) $\displaystyle\int_0^{2\pi} \frac{dx}{2+\cos x}$

7 Feynman's Trick

$$I(\tau) = \int_a^b f(x,\tau)\,dx = \int\left(\int_a^b \frac{\partial}{\partial t}f(x,t)\,dx\right)dt\Big|_{t=\tau} \qquad \text{for } f\in C^1([a,b]) \tag{7.1}$$

Proof. If the function $f(x,t)$ is continuously differentiable in both arguments, the order of differentiation and integration can be switched:

$$\frac{d}{dt}\int_a^b f(x,t)\,dx = \int_a^b \frac{\partial}{\partial t}f(x,t)\,dx \tag{7.2}$$

The formula for the Feynman Trick then follows by integrating back and evaluating at $t=\tau$. □

Example 7.1

$$\int_0^1 \frac{x^2-1}{\ln x}\,dx \tag{7.3}$$

First we need to define the integrand as a 2 parameter function. For this we essentially can put our parameter t anywhere, as long as the function is continuously differentiable. When looking at our integral we can see that it would be good to get rid of the logarithm in the denominator. To get a logarithm after differentiation we will use the fact that

$$\frac{d}{dt}x^t = x^t \ln x \tag{7.4}$$

With this knowledge we will write our integral as

$$I(t) = \int_0^1 \frac{x^t - 1}{\ln x}\, dx \tag{7.5}$$

In which case the original integral would be $I(2)$ By differentiating the integral with respect to t we get

$$I'(t) = \frac{d}{dt}I(t) = \int_0^1 \frac{\partial}{\partial t}\frac{x^t - 1}{\ln x}\, dx = \int_0^1 \frac{x^t \ln x}{\ln x}\, dx = \int_0^1 x^t\, dx = \frac{1}{1+t} \tag{7.6}$$

To get back to our integral $I(t)$ we need to integrate again, this time with respect to t:

$$I(t) = \int \frac{1}{1+t}\, dt = \ln(1+t) + C \tag{7.7}$$

Here, it is import to consider a constant factor C added to the indefinite integral. Because our original integral was definite, integrating it directly would not yield a constant. We therefore still need to determine this constant C. We do this by trying to calculate the integral $I(t)$ directly for a t different from our starting $t = 2$. In particular, evaluating the integral at $t = 0$ gives

$$I(0) = \int_0^1 \frac{x^0 - 1}{\ln x}\, dx = \int_0^1 \frac{0}{\ln x}\, dx = 0 \tag{7.8}$$

The expression (7.7) for the integral should give the same result:

$$I(0) = \ln(1+0) + C = C \tag{7.9}$$

Equating both results tells us that the constant is $C = 0$. Our original integral can then be calculated by inserting $t = 2$ in (7.7)

$$I(2) = \int_0^1 \frac{x^2 - 1}{\ln x}\, dx = \ln(2+1) + 0 = \ln 3 \tag{7.10}$$

Example 7.2

$$\int_0^\infty \frac{\sin x}{x}\, dx \tag{7.11}$$

For this integral we again need to find a t-dependency that allows us to integrate a simpler function after differentiation. To remove the x from the denominator, we introduce a factor e^{-tx} to the integrand.

$$I(t) = \int_0^\infty \frac{\sin x}{x}\, e^{-tx}\, dx \tag{7.12}$$

With this, the desired integral will be $I(0)$. Differentiating results in

$$I'(t) = \int_0^\infty \frac{\partial}{\partial t}\left(\frac{\sin}{x}\, e^{-tx}\right)\, dx = \int_0^\infty \frac{\sin x}{x}(-tx)\, e^{-tx}\, dx = -\int_0^\infty \sin x\, e^{-tx} \tag{7.13}$$

Integrating by parts twice gives

$$\begin{aligned} I'(t) &= -\left(-\cos x\, e^{-tx}\Big|_0^\infty - t\int_0^\infty \cos x\, e^{-tx} \right) \\ &= (0-1) + t\left(\sin x\, e^{-tx}\Big|_0^\infty + t\int_0^\infty \cos x\, e^{-tx}\, dx \right) \\ &= -1 + t\cdot(0-0) - t^2\cdot I'(t) \end{aligned} \tag{7.14}$$

$$\implies \quad I'(t) = -\frac{1}{1+t^2} \tag{7.15}$$

Now we once again integrate with respect to t to get

$$I(t) = -\int \frac{1}{1+t^2}\, dt = -\arctan t + C \tag{7.16}$$

To find the constant C we see that $I(t)$ tends to 0 as $t \to \infty$. Simultaneously $\arctan t \to \frac{\pi}{2}$, which implies that $C = \frac{\pi}{2}$ for (7.16) to also go to 0. Our integral from the start therefore is

$$I(0) = \int_0^\infty \frac{\sin x}{x}\, dx = -\arctan 0 + \frac{\pi}{2} = \frac{\pi}{2} \tag{7.17}$$

8 Reduction Formulas

$$I_n = \int f_n(x)\, dx = g(x, n, I_{n-1}, I_{n-2}, \ldots) \tag{8.1}$$

Notable reduction formulas

$$I_n = \int \cos^n x\, dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2} \tag{8.2}$$

$$I_n = \int \sin^n x\, dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n} I_{n-2} \tag{8.3}$$

$$I_n = \int x^n e^{ax}\, dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1} \tag{8.4}$$

Example 8.1

$$I_n = \int \cos^n x\, dx \tag{8.5}$$

Integrating by parts with $u = \cos^{n-1} x$ and $dv = \cos x\, dx$ gives

$$I_n = \cos^{n-1} x \sin x - \int \sin x(-(n-1)\cos^{n-2} x \sin x)\, dx \tag{8.6}$$

$$= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x\, dx \tag{8.7}$$

Now we can use the identity

$$\sin^2 x = 1 - \cos^2 x \tag{8.8}$$

which results in

$$I_n = \cos^{n-1} x \sin x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x \, dx \tag{8.9}$$

$$= \cos^{n-1} x \sin x + (n-1) \left(\int \cos^{n-2} x \, dx - \int \cos^n x \, dx \right) \tag{8.10}$$

$$= \cos^{n-1} x \sin x + (n-1)(I_{n-2} - I_n) \tag{8.11}$$

Now we can solve for integral we are looking for:

$$I_n + (n-1) I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2} \tag{8.12}$$

$$\implies \quad I_n = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2} \tag{8.13}$$

Practice Integrals:

a) $\displaystyle\int \frac{x^n}{\sqrt{ax+b}} \, dx$

b) $\displaystyle\int \frac{1}{(x^2+1)^n} \, dx$

c) $\displaystyle\int \sec^n x \, dx$

9 Power Series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \quad \iff \quad \int f(x) \, dx = \sum_{n=0}^{\infty} \frac{a_n (x-c)^{n+1}}{n+1} + \text{constant} \tag{9.1}$$

Notable power series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{9.2}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \tag{9.3}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \tag{9.4}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad |x| < 1 \tag{9.5}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \qquad -1 < x \le 1 \tag{9.6}$$

10 Floor Function Integrals

$$\int_m^n f(\lfloor x \rfloor)\,dx = \sum_{k=m}^{n-1} f(k) \qquad m < n \in \mathbb{N} \tag{10.1}$$

$$\int_a^b f(\lfloor x \rfloor)\,dx = (b - \lfloor b \rfloor) \cdot f(\lfloor b \rfloor) + (\lceil a \rceil - a) \cdot f(\lfloor a \rfloor) + \int_{\lceil a \rceil}^{\lfloor b \rfloor} f(\lfloor x \rfloor)\,dx \qquad a, b \in \mathbb{R} \tag{10.2}$$

11 Additional Skills

11.1 Polynomial Long Division

11.2 Partial Fraction Decomposition

11.3 Trigonometric/Hyperbolic Identities

$$\sin^2 x + \cos^2 x = 1 \qquad \cosh^2 x - \sinh^2 x = 1$$

$$\sec x = \frac{1}{\cos x}$$

$$\csc x = \frac{1}{\sin x}$$

$$\tan x = \frac{\sin x}{\cos x} \qquad \tanh x = \frac{\sinh x}{\cosh x}$$

$$\cot x = \frac{\cos x}{\sin x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \qquad \sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y \qquad \cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y} \qquad \tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

$$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot x \pm \cot y} \qquad \coth(x \pm y) = \frac{\coth x \coth y \pm 1}{\coth x \pm \coth y}$$

$$\sin x \pm \sin y = 2 \sin\left(\frac{x \pm y}{2}\right) \cos\left(\frac{x \mp y}{2}\right) \qquad \sinh x \pm \sinh y = 2 \sinh\left(\frac{x \pm y}{2}\right) \cosh\left(\frac{x \mp y}{2}\right)$$

$$\cos x + \cos y = 2 \cos\left(\frac{x + y}{2}\right) \cos\left(\frac{x - y}{2}\right) \qquad \cosh x + \cosh y = 2 \cosh\left(\frac{x + y}{2}\right) \cosh\left(\frac{x - y}{2}\right)$$

$$\cos x - \cos y = -2 \sin\left(\frac{x + y}{2}\right) \sin\left(\frac{x - y}{2}\right) \qquad \cosh x - \cosh y = 2 \sinh\left(\frac{x + y}{2}\right) \sinh\left(\frac{x - y}{2}\right)$$

$$\tan x \pm \tan y = \frac{\sin(x \pm y)}{\cos x \cos y} \qquad \tanh x \pm \tanh y = \frac{\sinh(x \pm y)}{\cosh x \cosh y}$$

$$\cot x \pm \cot y = \frac{\sin(x \pm y)}{\sin x \sin y} \qquad \coth x \pm \coth y = \frac{\sinh(y \pm x)}{\sinh x \sinh y}$$

$$\sin x \sin y = \frac{\cos(x - y) - \cos(x + y)}{2} \qquad \sinh x \sinh y = \frac{\cosh(x + y) - \cosh(x - y)}{2}$$

$$\cos x \cos y = \frac{\cos(x - y) + \cos(x + y)}{2} \qquad \cosh x \cosh y = \frac{\cosh(x + y) + \cosh(x - y)}{2}$$

$$\sin x \cos y = \frac{\sin(x - y) + \sin(x + y)}{2} \qquad \sinh x \cosh y = \frac{\sinh(x + y) + \sinh(x - y)}{2}$$

$$\sin(2x) = 2\sin x\cos x \qquad \sinh(2x) = 2\sinh x\cosh x$$
$$\cos(2x) = \cos^2 x - \sin^2 x \qquad \cosh(2x) = \cosh^2 x + \sinh^2 x$$
$$\tan(2x) = \frac{2\tan x}{1-\tan^2 x} \qquad \tanh(2x) = \frac{2\tanh x}{1+\tanh^2 x}$$
$$\cot(2x) = \frac{\cot^2 x - 1}{2\cot x} \qquad \coth(2x) = \frac{\coth^2 x + 1}{2\coth x}$$

$$\sin\tfrac{x}{2} = \sqrt{\frac{1-\cos x}{2}} \qquad \sinh\tfrac{x}{2} = \sqrt{\frac{\cosh x - 1}{2}}$$
$$\cos\tfrac{x}{2} = \sqrt{\frac{1+\cos x}{2}} \qquad \cosh\tfrac{x}{2} = \sqrt{\frac{\cosh x + 1}{2}}$$
$$\tan\tfrac{x}{2} = \frac{\sin x}{1+\cos x} \qquad \tanh\tfrac{x}{2} = \frac{\sinh x}{\cosh x + 1}$$
$$\cot\tfrac{x}{2} = \frac{\sin x}{1-\cos x} \qquad \coth\tfrac{x}{2} = \frac{\sinh x}{\cosh x - 1}$$

$$\sin^2 x = \frac{1-\cos(2x)}{2} \qquad \sinh^2 x = \frac{\cosh(2x)-1}{2}$$
$$\cos^2 x = \frac{1+\cos(2x)}{2} \qquad \cosh^2 x = \frac{\cosh(2x)+1}{2}$$
$$\tan^2 x = \frac{\sin^2(2x)}{(1+\cos(2x))^2} \qquad \tanh^2 x = \frac{\sinh^2(2x)}{(\cosh(2x)+1)^2}$$
$$\cot^2 x = \frac{\sin^2(2x)}{(1-\cos(2x))^2} \qquad \coth^2 x = \frac{\sin^2(2x)}{(\cosh(2x)-1)^2}$$

$$\sin(\arctan x) = \frac{x}{\sqrt{1+x^2}} \qquad \sinh(\operatorname{artanh} x) = \frac{x}{\sqrt{1-x^2}}$$
$$\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}} \qquad \cosh(\operatorname{artanh} x) = \frac{1}{\sqrt{1-x^2}}$$
$$\tan(\arcsin x) = \frac{x}{\sqrt{1-x^2}} \qquad \tanh(\operatorname{arsinh} x) = \frac{x}{\sqrt{x^2+1}}$$
$$\tan(\arccos x) = \frac{\sqrt{1-x^2}}{x} \qquad \tanh(\operatorname{arcosh} x) = \frac{\sqrt{x^2-1}}{x}$$

$$\arctan x = \operatorname{arccot}\tfrac{1}{x} \qquad \operatorname{artanh} x = \operatorname{arcoth}\tfrac{1}{x}$$
$$\arcsin x + \arccos x = \frac{\pi}{2}$$
$$\arctan x + \operatorname{arccot} x = \frac{\pi}{2}$$

$$\sin x = -i\sinh(ix) \qquad \sinh x = -i\sin(ix)$$
$$\cos x = \cosh(ix) \qquad \cosh x = \cos(ix)$$
$$\tan x = -i\tanh(ix) \qquad \tanh(x) = -i\tan(ix)$$

$$\arcsin x = -i\operatorname{arsinh}(ix) \qquad \operatorname{arsinh} x = -i\arcsin(ix)$$
$$\arccos x = \pm i\operatorname{arcosh} x \qquad \operatorname{arcosh} x = \pm i\arccos x$$
$$\arctan x = -i\operatorname{artanh}(ix) \qquad \operatorname{artanh} x = -i\arctan(ix)$$

Practice Integrals:

a) $\int \sqrt{a^2 + x^2}\, dx$

b) $\int \frac{\tan^2 x}{\sin(2x)}\, dx$

c) $\int \frac{x^2}{\sqrt{x^2 + 25}}\, dx$

d) $\int_{-\pi-1}^{\pi+1} \sin x \sinh x\, dx$

12 More Practice Integrals

a) $\displaystyle\int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx$

b) $\displaystyle\int_0^\pi \ln(\sin(x))\,dx$

c) $\displaystyle\int_0^\infty \frac{1}{(1+x^a)(1+x^2)}\,dx$

d) $\displaystyle\int_{-\frac{\pi}{2}}^\pi x^7 \sin x\,dx$

e) $\displaystyle\int \sin^n x \cos^m x\,dx$

f) $\displaystyle\int \frac{1}{x^2-x+1}\,dx$