ANSWER BOOKLET

PLANCKS
London 2020
0th online edition

This PLANCKS competition contains 15 pages (including this cover page) and 9 problems. Please work on paper and put both your team's name and question number on the top of every page.

You are required to show your work on each problem on this exam. The following rules apply:

- The contest consists of **9 problems, each worth 10 points**. Subdivisions may be indicated in the problems.
- **Organize your work**, in a reasonably neat and coherent way. All problems must be done on paper, in English, and scans uploaded before the end of the submission period into your team folder, which can be found at www.plancks.uk/exam. Files must be named appropriately, with your team name and question number and page number on every page. **Please upload the answer to each question as a separate file**.
- When a problem is unclear, a participant can ask, via the zoom call, for a clarification. If the response is relevant to all teams, the jury will provide this information to the other teams.
- The use of hardware (including phones, tablets, etc.) is not approved, except of scientific, non programmable calculators, watches and medical equipment. The use of computers should be limited to receiving/uploading the questions and communicating with your team and the jury. **Internet resources should not be used to answer questions**.
- The organisation has the right to disqualify teams at any point for misbehaviour or breaking the rules.
- In situations to which no rule applies, the organisation decides.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
Total:	90	

This answer booklet contains 37 pages. The question number is given in the top left corner. Other question or page numbers referenced may not be correct in this combined document.

Q1

A snow plough problem
Professor Alan Davies

Question

On a snowy winter's morning a snow plough leaves the yard at 06.00 precisely and makes its way along a long straight road. The driver notices that in the second hour she has travelled half the distance that she did in the first hour. At what time did it start snowing?

State a small set of reasonable assumptions which lead to the following set of variables and parameters:

Suitable variables and parameters
Time of travel t, with $t = 0$ at 06.00
Distance travelled by plough $x(t)$
Height of snow $h(t)$
Rate of snowfall per unit area R_S
Rate of snow removal by plough R_P
Width of snow plough blade w
Time snow is falling before plough starts T

All measured in SI units

Obtain:

1. a relationship between w, h, R_P, Δx and Δt, where the plough moves a short distance, Δx, in a time interval Δt
2. a relationship between h, R_S, t and T

Q1

Solution

Some reasonable assumptions

The snowfall rate is constant.
The snow plough removes snow at a constant rate.
The height of the plough's blade is greater than the depth of the snow.

In a small interval of time, Δt, the snow plough moves a distance Δx, see figure 1

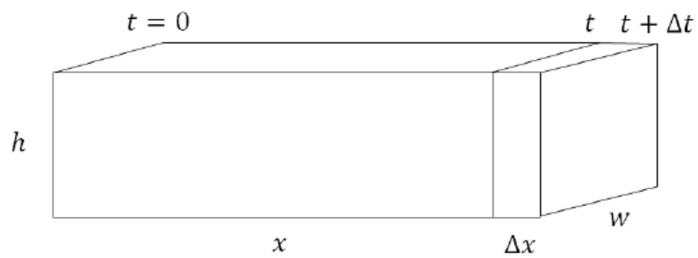


Figure 1: Snow depth at time t.

The volume of snow, $wh\Delta x$, removed in that time is $R_P \Delta t$. Which leads to

$$wh\Delta x = R_P \Delta t \tag{1}$$

At time t total volume of snow removed is xwh. This had fallen in time $t+T$ and is also given by $R_S(t+T)wx$.
Hence

$$h = R_S(t+T)wx \tag{2}$$

From equations (1) and (2) we have

$$R_S(t+T)w\Delta x = R_P \Delta t$$

Taking the limit as $\Delta x, \Delta t \to 0$ we obtain the differential equation

$$\frac{dx}{dt} = \alpha \frac{1}{t+T} \tag{3}$$

where $\alpha = \frac{1}{w}\frac{R_P}{R_S}$.

Solving the differential equation (3)

$$x = \alpha \left[\ln(t+T) + \ln C\right]$$

We find the arbitrary constant using the initial condition $x(0) = 0$:

$$0 = \alpha \left[\ln(T) + \ln C\right]$$

so that $\ln C = -\ln(T)$ and hence

$$x = \alpha \ln\left(1 + \frac{t}{T}\right) \tag{4}$$

So far our units for time are seconds, not particularly useful in this problem. So we switch to time in hours. We see from equation (3) that α is independent of time and that every term in equation (4) is independent of time explicitly hence we may use any unit without changing the equation.

Suppose that the snow plough travels a distance d in the first hour, then from equation (4)

$$d = \alpha \ln\left(1 + \frac{1}{T}\right)$$

and since the distance travelld in the second hour is half that in the first hour

$$\frac{3}{2}d = \alpha \ln\left(1 + \frac{2}{T}\right)$$

Hence

$$2\ln\left(1 + \frac{2}{T}\right) = 3\ln\left(1 + \frac{1}{T}\right)$$

and

$$\left(1 + \frac{2}{T}\right)^2 = \left(1 + \frac{1}{T}\right)^3$$

After a little algebra this leads to the quadratic equation

$$T^2 - T - 1 = 0$$

With solution

$$T = \frac{-1 \pm \sqrt{5}}{2}$$

The negative sign is not appropriate so we have $T = \frac{-1+\sqrt{5}}{2} \approx 0.618034$
Which is 37 minutes 5 seconds

Hence, it started to snow at precisely 05:22:55

Q2

Question:

In the toddler game 'Dotty Dinosaur', there is a fair dice with 6 colours. After rolling a colour, you collect a dot for your dinosaur of that colour – if you have already got a dot of that colour you don't do anything. You 'win' and the game ends when you have collected all six dots, one of each colour.

Calculate the mean number of throws of the dice needed to 'win' a game of dotty dinosaur.

Q2

Solution: I know of three solutions to this problem. Two of them are standard but long – one using a generating function and the second using a transfer matrix. I will write these up when I get time. There is also a very elegant solution if you can find it which I describe here.

Let e_k be the expected number (i.e. mean) of further throws needed to 'collect' all six colours when you already have k colours. Clearly, $e_6 = 0$ as if you already have all of the colours you don't need to throw again. The answer to the question is e_0, i.e. the number of throws needed to collect 6 colours when you don't have any of them yet.

We will work backwards from e_6. If you already have 5 dots, then there is a 1/6 chance you will get the final colour and have 6 dots, and a 5/6 chance you will remain on 5 dots. This gives a relationship between expectation values:

$$e_5 = 1 + \frac{1}{6}e_6 + \frac{5}{6}e_5.$$

This is interpreted as the number of further throws you need if you have five dots is:

- One throw (the one you have just done),
- Plus a 1/6 chance you get to six dots, in which case you need a further e_6 throws,
- Plus a 5/6 chance you remain on five dots, in which case you need a further e_5 throws.

This can be worked down:

$$\begin{aligned}
e_5 &= 1 + \frac{1}{6}e_6 + \frac{5}{6}e_5 \\
e_4 &= 1 + \frac{2}{6}e_5 + \frac{4}{6}e_4 \\
e_3 &= 1 + \frac{3}{6}e_4 + \frac{3}{6}e_3 \\
e_2 &= 1 + \frac{4}{6}e_3 + \frac{2}{6}e_2 \\
e_1 &= 1 + \frac{5}{6}e_2 + \frac{1}{6}e_1 \\
e_0 &= 1 + \frac{6}{6}e_1
\end{aligned}$$

One could immediately start at the top substituting in $e_6 = 0$ and working down to get the answer. However it is quite nice to look at the structure if one multiplies each equation by 6 and subtracts the third term on the RHS from each of them:

$$\begin{aligned}
e_5 &= 6 + e_6 \\
2e_4 &= 6 + 2e_5 \\
3e_3 &= 6 + 3e_4 \\
4e_2 &= 6 + 4e_3 \\
5e_1 &= 6 + 5e_2 \\
6e_0 &= 6 + 6e_1
\end{aligned}$$

Now, dividing the second one by 2, the third one by 3 and so on gives us a recurrence relation:

$$\begin{aligned} e_5 &= \frac{6}{1} + e_6 \\ e_4 &= \frac{6}{2} + e_5 \\ e_3 &= \frac{6}{3} + e_4 \\ e_2 &= \frac{6}{4} + e_3 \\ e_1 &= \frac{6}{5} + e_2 \\ e_0 &= \frac{6}{6} + e_1 \end{aligned}$$

Finally, putting in $e_6 = 0$ gives us the rather nice formula:

$$e_0 = 6\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\right) = \frac{147}{10} = 14.7 \tag{1}$$

Thus on average, the game will last 14.7 throws.

This is a very nice (but surprisingly little known) formula for 'collecting' N objects, if you randomly get 1 of the N each time. On average, you will need

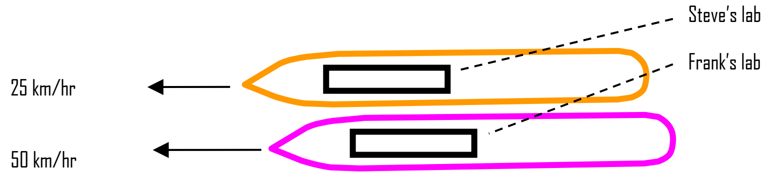
$$E = N \sum_{i=1} N \frac{1}{i} \sim N \ln N \tag{2}$$

collections to make sure you have at least one of each item. In other words, you will need on average $\ln N$ of each item to get them all.

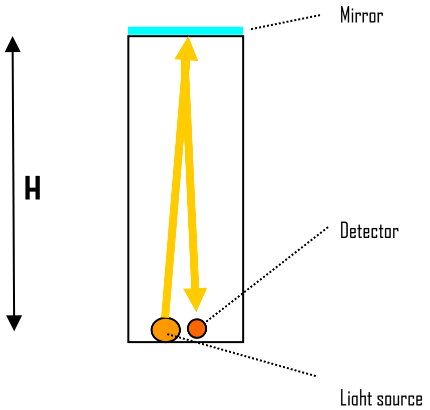
Q3

A STORY OF ABOUT THE TIME OF EINSTEIN !

Suppose that two experimenters, Frank and Steve, are carrying out identical measurements using light on board two liners that are crossing the Atlantic in calm weather. Frank has his laboratory on the deck of one liner and Steve has his laboratory on the deck of the other liner. The liner with Frank on board is travelling at 50 km/hr and the liner with Steve on board is travelling at 25 km/hr in the same direction. Both laboratories have large windows so Frank can look into Steve's lab from his own lab and vice versa for the period that the liners are side by side, with the faster overtaking the slower:



Both experimenters each have an identical piece of equipment:



A light source sends a pulse of light vertically upwards to be then reflected vertically downwards by a mirror. A highly accurate detector measures the time for the up-down journey of the pulse.

Both Frank and Steve carry out the same experiment each in their own lab and calculate the speed of light ***C*** by measuring the time of flight ***T*** of the pulse:

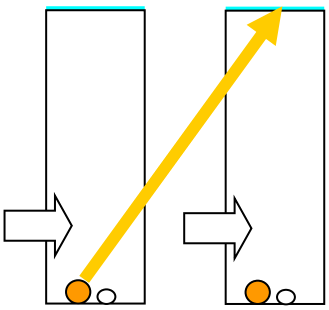
$$C = \frac{2H}{T} \qquad \text{...... Equation 1}$$

They talk by phone and find that each has found that their two results for the speed of light ***C*** are exactly the same. They are not surprised by this because, although one lab was moving at 25 km/hr relative to the other, neither lab was accelerating. Specifically, these experimenters are remembering from their university physics lectures that:

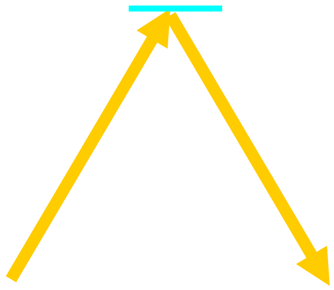
The 'laws' of physics are the same for all experimenters in non-accelerating motion

Q3

They now carry out a second experiment by adjusting each detector so that it can measure the time of flight of the pulse in the other laboratory. Frank's liner is overtaking Steve's and Frank sees Steve and his lab and his equipment moving to his (Frank's) right:



Frank sees the pulse of light in Steve's equipment start its upward journey and sees Steve's mirror move to the right during the time taken for the pulse to reach the mirror. After reflection, Frank sees the pulse following an angled path downwards as shown in this diagram:



Steve sees Frank and his lab moving to his right as Frank's liner overtakes. When they compare notes, both Steve and Frank find that they have observed the same angled-path phenomenon.

They now do some analysis and write down the following:

*Let **T'** be the time taken for the pulse to travel from the source and back to the detector as measured by the experimenter on board the other liner. During this time the relative movement of one liner relative to the other* AB *is **VT'** where **V** is the speed of one liner relative to the other.*

.

*The distance **D** travelled by the pulse according to the experimenter in the other liner is:*

$$D = 2\sqrt{\left(\frac{VT'}{2}\right)^2 + H^2}$$

*Therefore, the speed of light **C'** according to the experimenter in the other liner, is* $\frac{D}{T'}$ *so:*

$$C' = \frac{2\sqrt{\left(\frac{VT'}{2}\right)^2 + H^2}}{T'} = \sqrt{V^2 + \frac{4H^2}{T'^2}} \qquad \text{.......... Equation 2}$$

Steve and Frank ponder on this result. They have recently heard that a guy called Einstein has been doing some work concerning the velocity of light but at this moment in time they are unaware of Einstein's conclusions. So Steve and Frank do the 'obvious' thing and agree that the **time** for the pulse of light to travel up and down vertically in one lab 'has to be exactly the same' as the pulse to travel up and down along the angled path in the other. That is, they assume that ***T = T'*** and combine Equations 1 and 2 to obtain:

$$C' = \sqrt{1 + \frac{V^2}{C^2}} \qquad \text{………… Equation 3}$$

This result they find most interesting, for it suggests that when an experimenter measures the speed of light C' in a laboratory that is moving relative to him, the measured speed is greater than that which he would measure in his own lab. The greater the difference in relative speed between the laboratories, the greater the difference between the two speeds of light.

Steve and Frank agree that their result is in fact a Earth-shaker and they get down to preparing a paper for publication.

They then manage to get hold of a copy of Einstein's 1905 paper and are disturbed by it. According to Einstein, every measurement of the speed of light (in a vacuum) – whatever the speed of the experimenter relative to the source of the light - will yield the **same** result. They can see that, if Einstein had been helping them with their experiments on board the two liners, then he would have argued that the speed of light as measured by Steve in his own lab and by Frank looking in on Steve's equipment in Steve's moving lab would be the **same**. The same result would be found by Steve looking in on Frank's experiment. Einstein would have argued that ***C=C'.***

Our two brave experimenters now make ***C=C'*** and combine Equations 1 and 2 and, after a bit of algebra, they obtain:

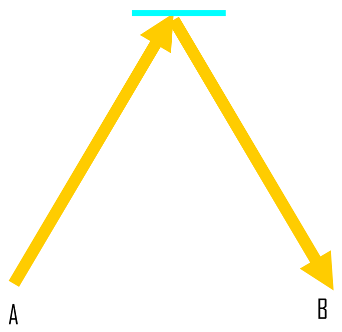
$$T' = \frac{T}{\sqrt{1 - \frac{V^2}{C^2}}} \qquad \text{…….. Equation 4}$$

Again, they ponder on what this means. There is no escaping from the fact that Equation 4 is telling them that, according to Frank looking in on Steve's moving lab, Steve's time ***T'*** was passing more slowly than his (Frank's) time ***T*** because the denominator of (4) will always be less than 1. And Steve would draw exactly the same conclusion about the rate of passing of Frank's time in Frank's moving lab. That the rate of passage of time is not the same for everyone is a more odd-ball result than that of the speed of light increasing when one looks in on a light-speed measurement experiment that is on the move !

Steve and Frank are faced with a choice. If they assume ***T = T'*** then they obtain Equation 3. If they assume ***C = C'*** then they obtain Equation 4. Which one is right ? They put the publishing of their epic paper on hold.!

Q3

Numbered Answers:
Note: similar answers can still be awarded points if they still make sense scientifically

1. $C = \frac{2H}{T}$
2. 

3. VT'
4. $D = 2\sqrt{\left(\frac{VT'}{2}\right)^2 + H^2}$
5. $\frac{D}{T'}$
6. $$C' = \frac{2\sqrt{\left(\frac{VT'}{2}\right)^2 + H^2}}{T'} = \sqrt{V^2 + \frac{4H^2}{T'^2}}$$
7. $C' = \sqrt{1 + \frac{V^2}{C^2}}$
8. **C=C'**
9. **C=C'**
10. $T' = \frac{T}{\sqrt{1 - \frac{V^2}{C^2}}}$

Q4

Time of Flight

Dr James Kneller - Queen Mary, University of London

(a) (2 points) Figure 2 shows a typical log-log plot of photocurrent against time. Deduce from this graph where the transit time will be and explain why you have chosen this position. You do not need to give an actual value of the transit time.

There are two linear regimes on this plot. The intersection of which will be the transit time, as displayed in the following figure, which is commonly referred to as the bending of the knee. This will be approximately $\approx 2.55\ \mu s$. The transit time is when the majority of carriers in the charge packet reaches the electrode, after this there is a sharp decrease in current.

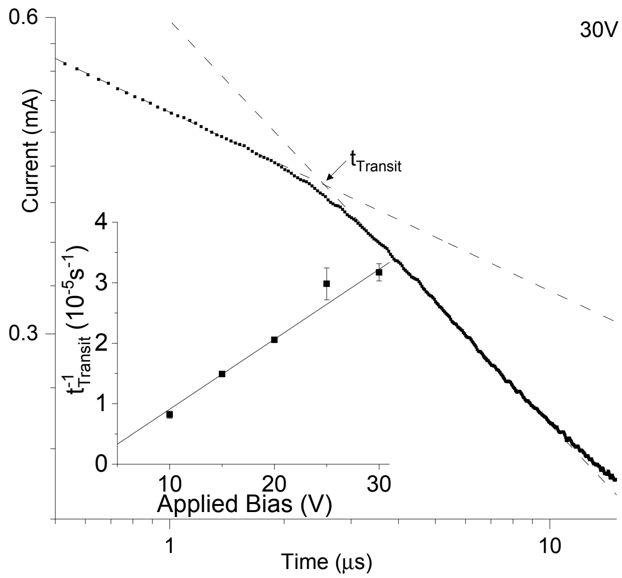


Figure 1: A sample displaying transient time of flight. With $h\nu$ denoting the incident photon and E the applied bias.

(b) (2 points) From figure 3, calculate the charge carrier mobility of this sample. Include an appropriate error in your result.

One needs to calculate the gradient of the line, as the mobility equation can be simplified to $\mu = md^2$. With this, the mobility calculates as $\mu = 2.05 \pm 0.14 \times 10^4\ cm^2V^{-1}s^{-1}$. Only the order of magnitude matters in mobility measurements, so the answer does not have to be exact. If there are no units or error do not award a mark.

(c) (2 points) We have not yet discussed what kind of charge carriers we are measuring. Using the information already provided, what charge particle have we measured mobility for? What happens to the opposite charge?

We are measuring holes. Fig 3 states that it's positive voltage being used to draw the carriers across which will mean positive holes *(1 mark)*. The electrons are *injected* into the ITO electrode which is grounded *(1 mark)*. Award a mark anyway if they mention excitons, which are bound electron hole pairs. This will form from the incident photon, and then the applied bias will separate the charges.

(d) (1 point) The charge carriers can get trapped while moving through the bulk of the sample on the way to the electrode. Give a example of one such kind of trap.

Electron traps. These are energy levels below the lowest-unoccupied-molecular-orbital (LUMO) which can cause the charge carriers to become immobile within the bulk. This increases the chances of recombination, above the usual Langevin rate (Coulomb attraction). The LUMO is the equivalent to the conduction band in non-organic structures. Do not award points for simply mentioning artefacts, but do award if they postulate a sound physics argument.

(e) (2 points) How will the charge particles being trapped affect the charge packet and how will figure 2 change?

Trapping causes the charge packet to break up and not move a a homologous unit *(1 mark)*. This causes dispersive transport of the charge carries (as the packet disperses). As the carriers are no longer arriving at the electrode at the same time, the sharp inflection peak observed in fig 2 will be obscured *(1 mark)*. The transits lines will also become curved.

(f) (1 point) What will figure 2 look like if we didn't apply a bias voltage? Explain why.

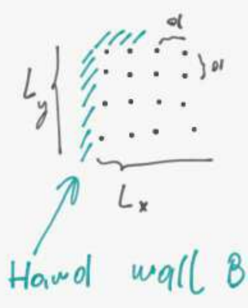
Without a bias voltage driving the charge carriers across we will still see a clear transit time in fig 2. This is due to there being a built-in bias caused from the unequal workfunctions of the two electrode materials. There will be less current being generated, but the shape of the graph should not change.

Q5 Phonon assisted charge transport in 2D

a) • Fermi Golden rule:

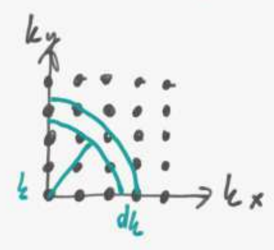
$$\Gamma_{A\to B} = \frac{2\pi}{\hbar} \left| \langle B,1 | (g c^+ |B\rangle\langle A|) | A,0\rangle \right|^2 \underbrace{\rho(\epsilon = qU)}_{\text{Density of states}} \quad (2)$$

• Calculation of DOS:



→ quantization of wavevector: $k_{x/y} \in \frac{\pi}{\underbrace{N_{x/y}a}_{L_{x,y}}} \{0, 1, 2, \dots, N_{x/y}-1\}$

k space:



$$\rho(k)dk = \frac{2\pi k}{4} dk \cdot \left(\frac{\pi}{L_x}\cdot\frac{\pi}{L_y}\right)^{-1} = \frac{S\,k\,dk}{2\pi}$$

$$\epsilon = \hbar v k$$

$$\rho(\epsilon) = \rho(k)\frac{dk}{d\epsilon} = \frac{\epsilon}{(\hbar v)^2}\cdot\frac{S}{2\pi} \quad (2)$$

$$\Rightarrow \Gamma_{A\to B} = \frac{2\pi}{\hbar}\cdot g^2 \cdot \frac{S}{2\pi}\frac{qU}{(\hbar v)^2} \quad \left[\frac{J^2}{J\cdot s}\cdot m^2 \cdot \frac{J}{J^2\cdot m^2} = \frac{1}{s}\right]$$

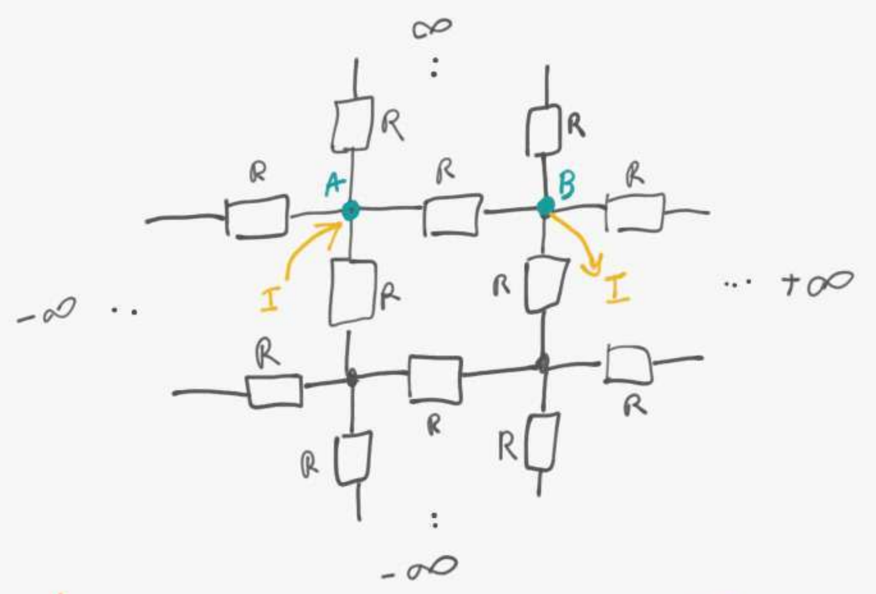
• Current: $I = q\Gamma_{A\to B} = \frac{q^2 g^2 S U}{\hbar^3 v^2}$

$\Rightarrow$ Resistance: $R = \frac{U}{I} = \frac{\hbar^3 v^2}{q^2 g^2 S}$ (1)

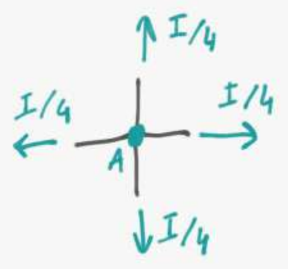
Q5 Phonon assisted charge transport in 2D

b) $R = \frac{\hbar^3 v^2}{q^2 g^2 S}$

$S \gg$ unit cell area $\Rightarrow \sim$ infinite grid.

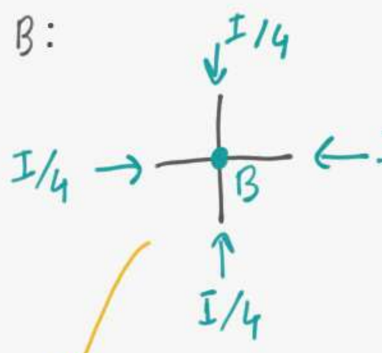


(1) Injecting I into A:



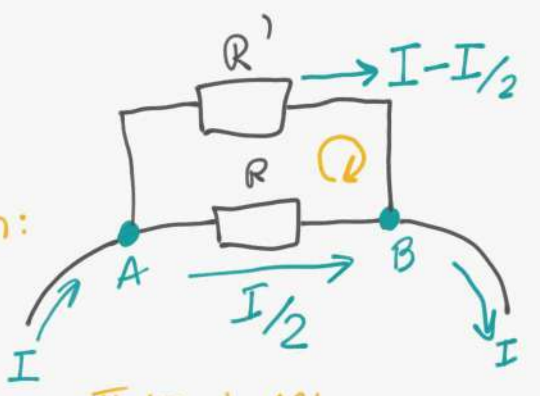
by symmetry.

injecting -I into B:



by symmetry

Superposition:



II Kirchoff's:

$R' I/2 - R I/2 = 0$

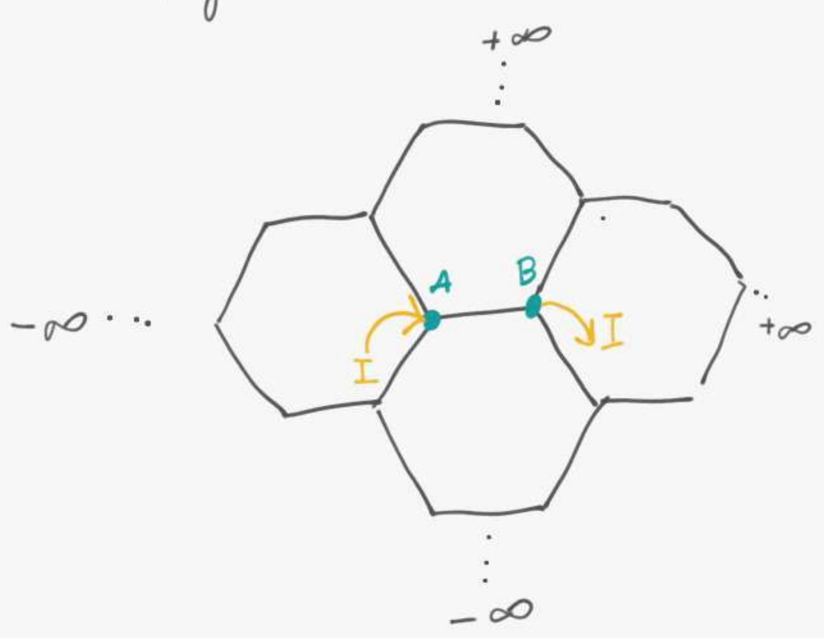
$\Rightarrow R = R'$ (1)

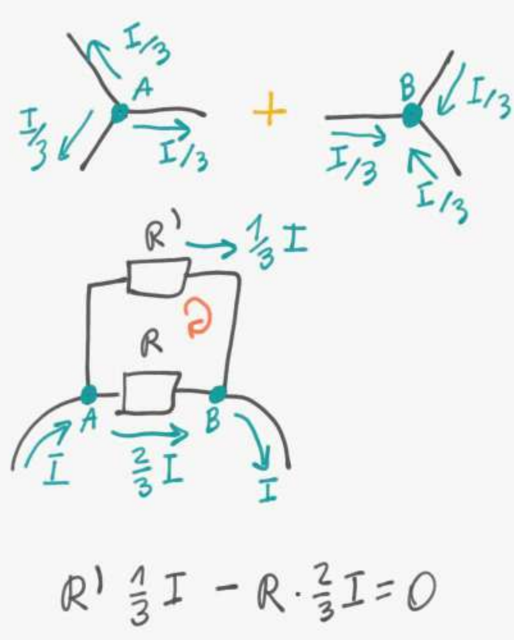
$\Rightarrow R_{AB} = \left(\frac{1}{R}+\frac{1}{R'}\right)^{-1} = R/2$ (1)

$\Rightarrow R_{AB} = \frac{\hbar^3 v^2}{2q^2 g^2 S}$

Q5 Phonon assisted charge transport in 2D

b) $R = \frac{\hbar^3 v^2}{q^2 g^2 S}$





$R' \frac{1}{3}I - R\cdot\frac{2}{3}I = 0$

$R' = 2R$ (1)

$R_{AB} = \left(\frac{1}{R}+\frac{1}{2R}\right)^{-1} = \frac{2}{3}R$

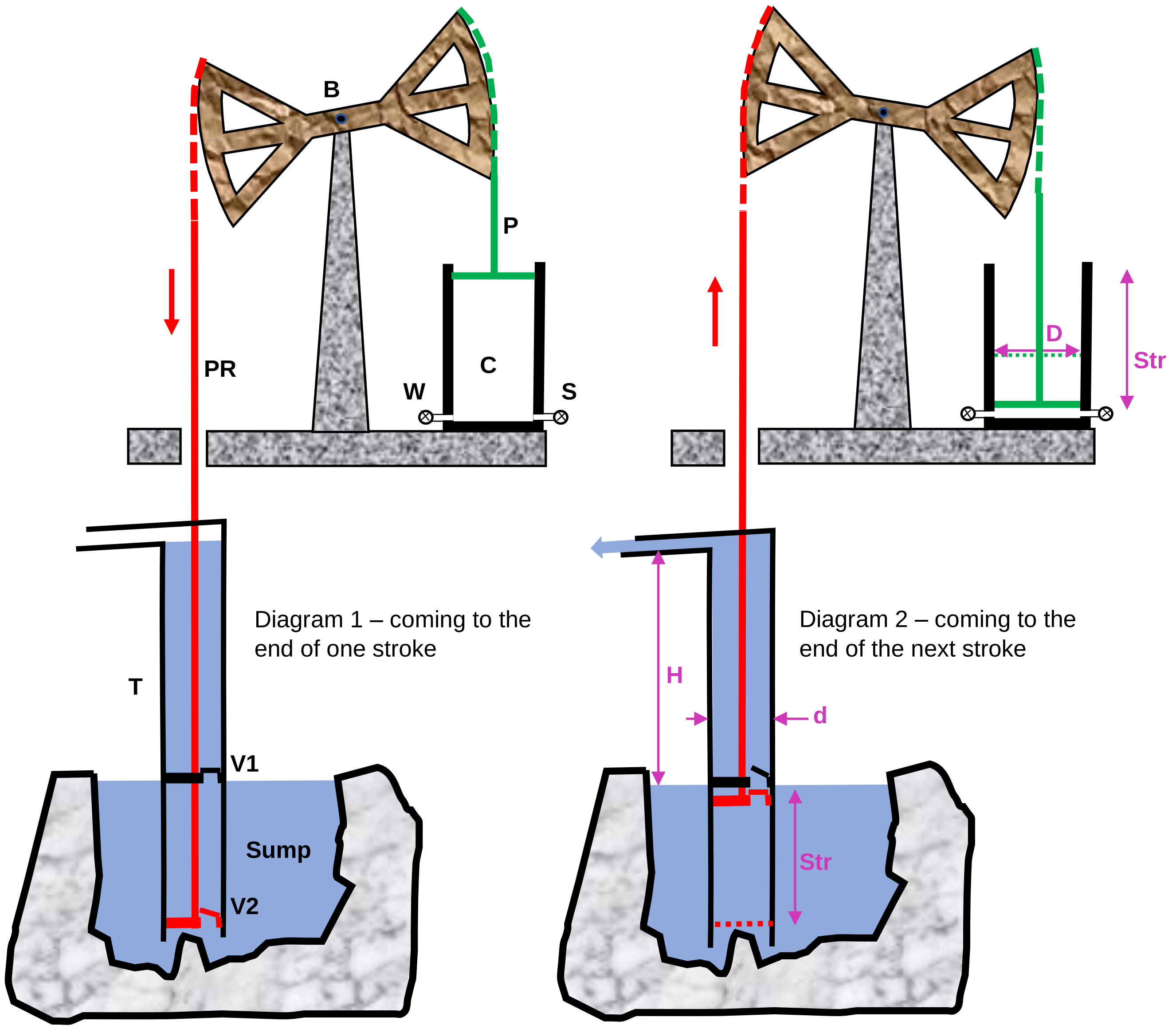
$R_{AB} = \frac{2}{3}\frac{\hbar^3 v^2}{q^2 g^2 S}$ (1)

Q6

Newcomen's Beam Engine

The Industrial Revolution in Britain started when 'steam engines 'began to supplement and then replace wind, water and horse power. The first of such engines was built by Thomas Newcomen in 1712 and was used to drain water from a coal mine that was about 50m deep. By the time of Newcomen's death in 1739 about a hundred of his engines were in operation in Britain. These were all beam engines. Their design was much improved by James Watt from 1760 onwards - with the oscillatory motion of beam engines being adapted in about 1780 to produce the more useful, rotary motion. All that will be now said is that during the next hundred and fifty years a huge range of steam engine variants emerged. There are several examples of Newcomen's Engine in existence, including one at the Science Museum, London.

Newcomen's original engine was massive, being about the size of a large house. Here is a hypothetical engine based roughly on Newcomen's design. It has the same operating principle of his engine but with quite different proportions appropriate for present purposes. The boiler supplying steam is not shown.



Q6

The large (wooden) beam **B** pivots about its mid-point. Its rocking motion raises and lowers a piston rod **P** and piston (green) in the 'steam 'cylinder **C** and a very long and heavy pump rod **PR** (red) that descends down the mine shaft. This latter rod moves within a fixed, cylindrical tube **T** that dips into the sump into which the mine water collects. **V1** is a 'flap 'valve fixed within **T**. **V2** is also a 'flap 'valve that is part of the piston attached to the end of the pumping rod. The weight of the (red) pumping rod **PR** and piston is intentionally greater than the weight of the (green) piston rod **P** and piston.

The original engine did not run by itself and required an operator to open and shut valves in sequence. Diagram 1 shows the engine just before the end of the pumping rod's downward motion which ends by the piston 'thumping 'against a 'stop'. Several cycles of engine operation have already taken place in order to prime the pump. The steam valve **S** has been open allowing steam at atmospheric pressure to enter cylinder **C**. The greater weight of the pumping rod has caused it to descend with valve **V2** open and, as the piston at the end of the rod moves down into the sump, sump water enters the lower part of the cylindrical tube **T**. Valve **V1** has remained shut as the pumping rod has moved downwards.

The operator now shuts off the steam supply valve **S** and for a very short time opens the valve **W** to introduce a fine spray of cold water into **C**. This condenses the steam and causes a vacuum to develop within **C**. Atmospheric pressure acting on the (green) piston begins to push it downwards and to begin to raise the pumping rod **PR**. Valve **V1** now opens and valve **V2** shuts. Diagram 2 shows the end of this stroke with the (green) piston about to 'bottom 'and stop. During the upward, 'working 'stroke of the engine water inside **T** is raised so as to flow out of the mine at ground level. The operator now opens the steam valve **S** to 'kill 'the vacuum in **C** and to start the next cycle of operation.

You are asked to calculate the volume of water raised up and removed from the sump per cycle of engine operation using the following information. A cycle is simply a downward and an upward stroke of **PR** or, equivalently, an upward and a downward stroke of **P**, with both strokes being of the same length. Not surprisingly, your answer will depend upon the assumptions that you make so it is important for you to make a list of these.

Excess mass of pumping rod PR over rod P, ExMass = 1200 kg

Stroke, Str = 2 m

Diameter, D = 0.8 m

Diameter, d = 0.3 m

Atmospheric pressure, AtmP = 0.1 MPa

Q6

Height, H = 50 m

Answer:

There will be valid variants to the following analysis:

It should be obvious that, during the 'working 'stroke, the force developed in cylinder **C** has to be sufficient to pull its end of the beam downwards so as to cause the other end of the beam pull **PR** upwards and raise a mass of water trapped inside tube **T**. The 'working 'stroke begins an instant after the engine state shown in Diagram 1.

Participants should make this observation of the system- 1 mark

With a vacuum 'instantaneously 'created inside **C** the downward force on the (green) piston is due to the atmospheric pressure acting on its upper surface.

Downward force on green piston $= \pi/4 \times D^2 \times AtmP$

$= \pi/4 \times 0.8^2 \times 100{,}000 = 50{,}265\ N$

Award a mark for calculating the downwards force on the green piston

At the instant the pumping rod **PR** begins to move upwards, valve **V2** slams shut and valve **V1** opens. The weight acting on the left-hand end of the beam is the excess weight of **PR** plus that of the column of water of length (***H + Str).***

Downward weight on red piston $= (\pi/4 \times d^2 \times (H + Str) \times Densw + ExMass) \times g$

$= (\pi/4 \times 0.3^2 \times (50 + 2) \times 1{,}000 + 1{,}200) \times 9.81\ N$

$= 47{,}830\ N$ .... ***(1)***

Award a mark for calculating the weight on the red piston

where $Densw = 1{,}000\ kg/m^3$ is the density of water and $g = 9.81\ m/s^2$ is the acceleration due to gravity.

Clearly, the two forces acting on the ends of the beam at the start of the stroke are unbalanced generating a moment so the beam will rotate increasingly more rapidly in a clockwise direction.

Participants should make this observation of the system- 1 mark

We have no information about the moment of inertia of beam **B** and, since the magnitudes of the above two forces are in fact very similar, we will assume that the beam rotates steadily about its centre so that both piston rods move at a steady, linear rate.

An assumption of a linear rate merits a mark (if participants make another assumption on the rate award also award mark making sure subsequent conclusions follow.)

As the pumping rod **PR** rises, the weight of water in **T** above the (red) piston within it gets less so we should investigate whether this is significant. Suppose at an instant between the engine states shown in Diagrams 1 and 2 the piston at the end of **PR** is at a depth **h** below the level of the water in the sump. The water pressure on the underside of this piston will be ***Densw g h*** and this will exert an upward force ***Densw g h A*** on the piston underside, where **A** is its area. The downward weight on the piston topside due to the column of water will be ***Densw g (H + h) A*** so the net downward weight will be ***Densw g H A.*** This means that the downward weight on the (red) piston does not change during the stroke and is as calculated above in ***(1).*** This is convenient !

Award a mark for discussion on how the weight of water will change and whether this change is significant.

During the working stroke the volume of water that is raised and flows out of **T** at its top is simply:

$$\textbf{\textit{Volume of water pumped out per cycle}} = Str \times A = 2 \times \pi/4 \times 0.3^2$$

$$\underline{= 0.141\ m^3}$$

Award a mark for the correct answer or one that follows from their previous assumptions although this really shouldn't make much of a difference.

Note it might be thought that this 'volume per cycle 'could be calculated immediately from inspection of Diagrams 1 and 2. To some extent this is true but it is vital to check that the operating forces are such to enable the engine to actually move through its intended operating cycle.

At the end of the 'working 'stroke the state of the engine is that shown in Diagram 2. The operator opens valve so as to admit steam into **C**. With no vacuum in **C** there are now no forces acting on the green piston. The excess weight of the (red) pumping rod **PR** will cause it to start to descend. This is a necessary feature of the engine. Again, unbalanced forces lead to accelerations so PR will accelerate downwards. Resistance to this motion will come from the viscosity of water flowing through valve **V2** but, since we have no information by which to calculate this resistance, we will simply assume that the downward motion of **PR** is steady.

A further assumption needed to be made is obvious one that sliding friction within the cylinder **C** and within tube **T** is negligible. Also, we will assume that the volume of 'spray 'water is negligible and does not build up within cylinder **C**.

A final comment of interest is that, soon after engines of the original design came into use, engineers arranged for the rocking movement of the beam to open and shut valves so there was no need for an operator. Newcomen engines now became self-acting and operated rather clunkily at a rate of about 12 strokes per minute.

Newcomen's engine is usually referred to as an 'atmospheric 'engine since it is atmospheric pressure rather that steam pressure that is used to do the work.

Award 3 marks for a discussion the system and assumptions made to solve the problem e.g. no friction in the cylinder etc.

Award 1 of these 3 marks if the participants discuss anything further from their conclusions e.g efficiency, how many stokes per minute etc.

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PLANCKS 2020 Finals
1888 Problem Solutions

We will accept answers based on the knowledge of participants if they were taking the exam in 1888 or using the knowledge of today. The solutions given to these questions are not entirely complete but markers should look for the key points indicated in **bold**.

Question 6- Part a

The Moon and the Earth are both heavy objects and are attracted to each other by gravity but why don't they fall into each other? Well, the Moon is in an orbit around the Earth meaning it has a velocity that is always perpendicular (**tangential velocity**) to the gravitational force pulling it towards the Earth. This keeps the balance between the pull towards and "push forwards" so to speak.

An example statement that would be awarded one mark- the moon is constantly falling, but its **velocity is tangential to gravity** means it will perpetually fall, but with a trajectory that never intercepts the surface. An object moving horizontally while experiencing a force in the vertical direction will have a curved trajectory. As long as the radius of that curve exceeds that of the Earth's own, the moon will orbit indefinitely, curving around the surface.

If participants don't give a full explanation and just briefly mention **tangential velocity** just award half a mark.

Question 6- Part b

The famous experiment refers to Archimedes work to find out whether the crown made for Hieron, the King of Syracuse, was made or pure gold or not as the goldsmith had claimed. This experiment has lead to one of the most well known scientific theories and led to the famous words uttered by every *mad scientist* in science fiction shouting "eureka" to their *discoveries*.

Archimedes principle- any body completely or partially submerged in a fluid at rest is acted upon by a buoyant force the magnitude of which is equal to the weight of the fluid displaced by the body. Award half a mark for stating the principle.

Archimedes principle can be used to find gravity by the relation given by Equation 1,

$$F_b = -\rho g V, \tag{1}$$

where F_b is the buoyant force, ρ is the density of the fluid, g is the acceleration due to gravity and V is the volume of the displaced fluid.

Q7

Participants can give a detailed description of an experiment involving a mass attached to a Newton meter being submerged into a fluid and recording the change on the Newton meter, then giving a suitable method for finding the volume and fluid density. Or any another equivalent method to be awarded half a mark.

Question 6- Part c

Award half a mark to answers sub-question.

(i) Gay-Lussac's law- pressure is proportional to the temperature with a fixed volume. If the **temperature is increased, so will the pressure**. A more full explanation is that the particles have been given more energy by the increase in temperature (they go nuts). Pressure is the force per unit area as the volume remains constant the area is also constant. The velocity of the particles has increased and so more collisions against the walls of the vessel and so the force and pressure both increase.

(ii) Charles's law- pressure multiplied by the volume of a fluid as constant temperature is itself a constant, **PV= constant**. This principle can be given by the example of squeezing a sealed fluid container like a balloon, the volume goes down and the pressure clearly goes up.

Question 6- Part d

Award half a mark for stating the following- Barometers are used to **measure atmospheric pressure**, above and below sea level. (It also can be used to measure altitude because of this fact).

Award half a mark for stating the following on the design of a mercury barometer (participants don't have to state that is is a mercury filled barometer it could be filled with another suitable fluid such as water)- **a tube filled partially with mercury is submerged upside down into a reservoir of mercury, the top of the tube contains a vacuum bubble. The tube has a scale on the side of it which indicates the rise and fall** in mercury which is due to the atmospheric pressure (see Figure 1). The principle works by balancing the weight of the mercury in the tube with the atmospheric pressure.

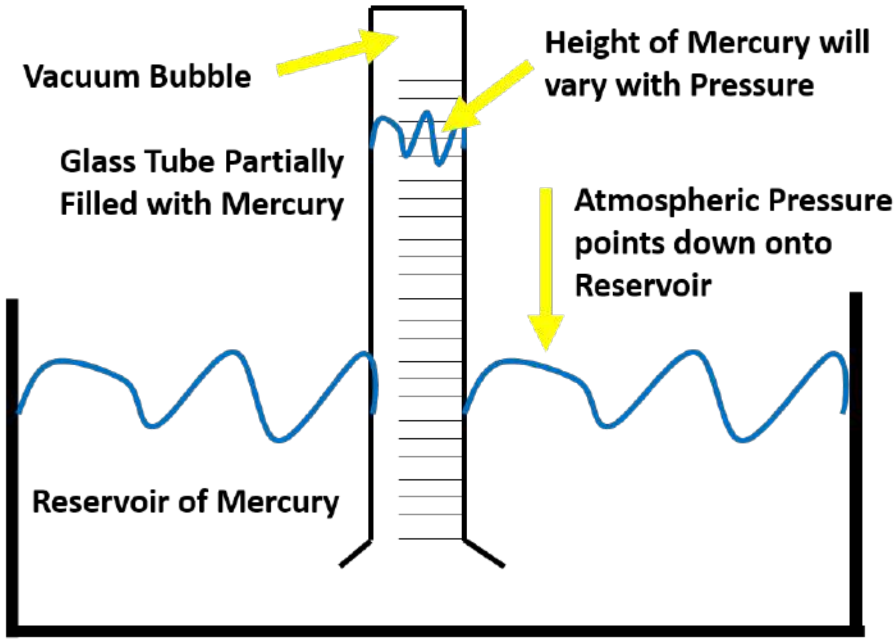

Figure 1: Diagram of a simple mercury based barometer.

Participant may write about the following explaining why the mercury level changes with the pressure, they can do so but it won't merit any further marks.- if the weight of mercury is less than the atmospheric pressure, the mercury level in the glass tube rises (high pressure). In areas of high pressure, air is sinking toward the surface of the earth more quickly than it can flow out to surrounding areas. Since the number of air molecules above the surface increases, there are more molecules to exert a force on that surface. With an increased weight of air above the reservoir, the mercury level rises to a higher level.

Question 6- Part e

Award half a mark for stating one the following explanation of what lightning is-

(i) Traditional- lightning is an electrical current which is generated when there is a build up of positive and negative charge in clouds. In clouds warm air causes the rain droplets to go up and cold air clouds containing ice crystals go down and they bump into each other. **The collisions create an eclectic charge. The positive charges (protons) form at the top of the cloud and the negative charges (electrons) form at the bottom of the cloud**. Since opposites attract, that causes a positive charge to build up on the ground beneath the cloud.

(ii) Modern- Relativistic runaway electron avalanche, growth of a population of relativistic electrons driven through a material (typically air) by an electric field.

(iii) Absurd- In Russia in the 1800s, when rain and thunder/ lightning were wanted three men climbed a tree. One would knock two fire-bands together; the sparks imitating lightning. Another would pour water over twigs, imitating rain. A third would bang on a kettle to attract thunder.

Award half a mark for stating the following on the suppose of lightning conductors- to **protect buildings and people**. Lightning rods provide a low-resistance path to ground that can be used to conduct the enormous electrical currents when lightning strikes occur. Award no marks for any references to the rod attracting the lightning to it.

Question 6- Part f

Award half a mark for a diagram similar to Figure 2.

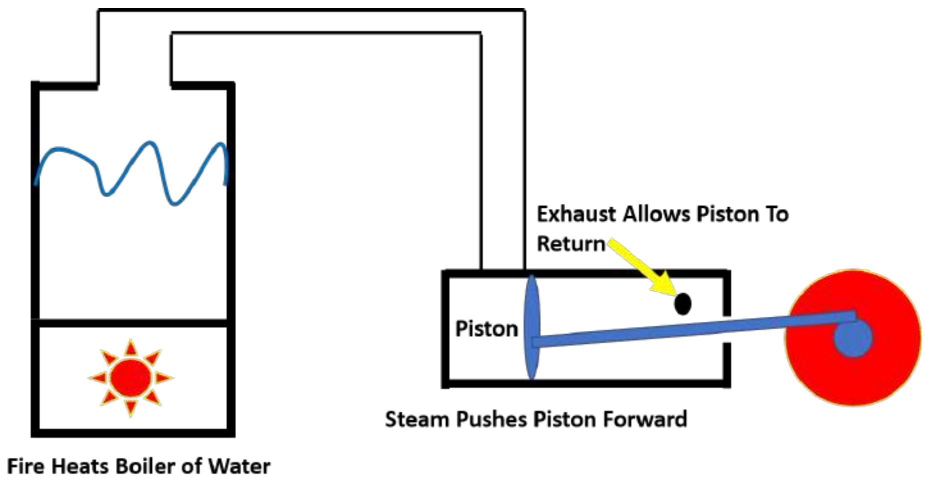

Figure 2: *Boil That Water*- A simple diagram with the key parts to a steam engine.

Award half a mark for explaining how this system work- water in a boiler is heated, steam is generated. The volume of water expands as it turns to steam inside the boiler, creating a high pressure. The **expansion of steam pushes the pistons**. At the end of the piston stroke, an exhaust port opens, allowing the steam to escape. Then the process is reversed and repeated in the opposite direction. The piston is connect to a driving wheel.

Question 6- Part g

Award half a mark for explaining what electrolysis is- the process by which **ionic substances are decomposed into constituent components when an electric current is passed through them**. Reference to the the increased rate of decomposition if the electrical energy is increased (increase in voltage). Participants can include a simple diagram shown in Figure 3.

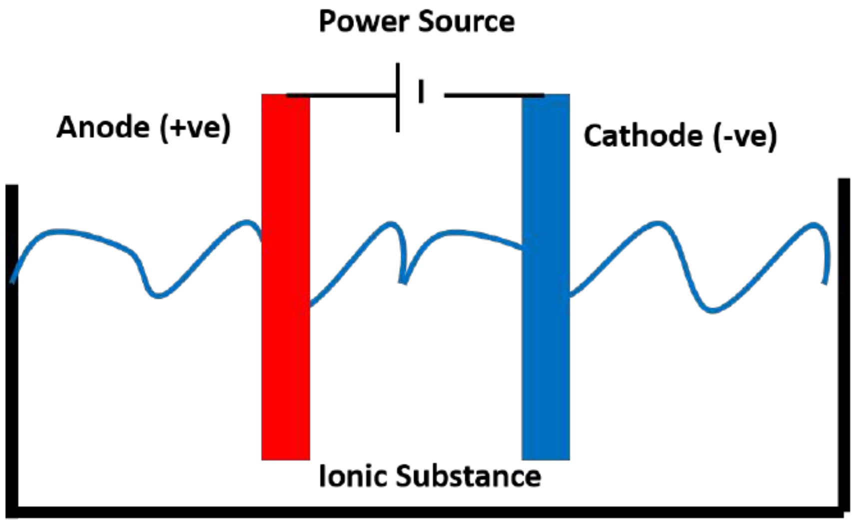

Figure 3: Diagram of the setup of an electrolysis experiment.

Award half a mark for an example of electrolysis- **water being separated into oxygen and hydrogen gas**, the electrolysis of brine produces hydrogen, chlorine gases and sodium hydroxide etc.

Question 6- Part h

Award half a mark for what a magnet is- A magnet is a material or object that **produces a magnetic field**, it has the property of attracting another magnet or a ferrous object. Participants may make some reference to a magnet being made up of magnetic moments which relate to the magnetic strength and orientation of a magnet or other object that produces a magnetic field.

Award half a mark for one of the methods of making a magnet-

(i) Magnetising some ferrous material- **running a piece of iron in the same direction over a magnet**, this makes all the magnetic ions point in the same direction as the field and when removed it has a permanent magnetic field due all all the moments being aligned.

(ii) Electromagnet- (Existed from 1824) **coil some insulated copper wire around an iron core and pass a current through it**. The loops of current will produce a magnetic field and magnetise the iron core. (Reference to Biot-Savart law).

Question 6- Part i

This question relates to photography in the 19th century. At the time of this paper there were several different types of camera and new advancements each decade we will accept a few different answers from the time.

The key point needed is that there is some material whether it be silver salts (commonly silver nitrate, sometimes silver iodine and silver chloride), gelatin emulsion or whatever they might give, that is **photo sensitive**. The salt or paste is coated on some paper or film and when exposed to sunlight it darkens. Objects blocking the light would leave shapes of the unaffected salt on the plate/ film. Award half a mark for discussion on how a photo-sensitive material darkens in sunlight.

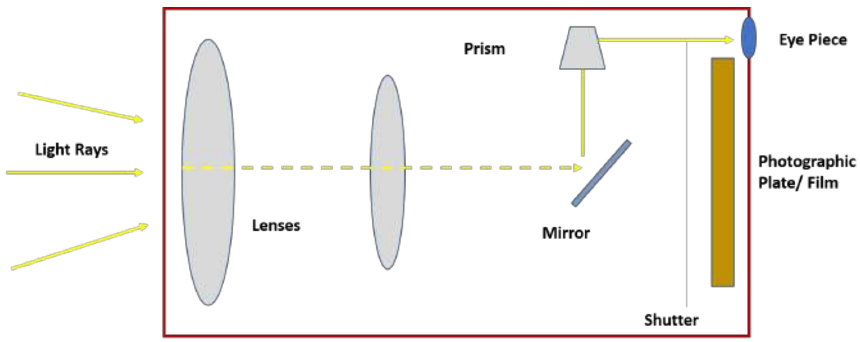

Figure 4: Diagram of the inner workings of a simple camera from the 1800s.

If a participant gives a diagram of a camera (as seen in Figure 4)but does not give a full discussion on the photo-sensitive material, award half a mark as an alternative.

The next part of the question asks why does photography fail when attempting to capture by gaslight. There are two ways of thinking about this an either will be accepted providing the participants provide sufficient reasoning as to why their choice is the true case. Award half a mark for a sufficient discussion on either case.

(i) Overexposure- having a gaslight near the camera creates an intense source of light that will completely saturate the photographic plate/ film and ruin the image.

(ii) Underexposure- the use of gaslight in the 1800s implies that it is either night time or they are indoors, so there is no sunlight to be used to take the image with. The gaslight does not provide sufficient light for the plate/ film to react and the image is very dull as a result.

Question 6- Part j

Participants need to prove

$$I(a,b) = \int_0^\infty e^{-a^2x^2}\cos(2bx)\,\mathrm{d}x = \frac{\sqrt{\pi}}{2a}e^{-\frac{b^2}{a^2}}. \tag{2}$$

Start with a substitution of $u = ax$, the limits of the integration will remain the same, which yields,

$$I(a,b) = \frac{1}{a}\int_0^\infty e^{-u^2}\cos\left(2\frac{b}{a}u\right)\mathrm{d}u, \tag{3}$$

letting $\omega = \frac{b}{a}$ then we can write,

$$I(\omega) = \frac{1}{a}\int_0^\infty e^{-u^2}\cos(2\omega u)\,\mathrm{d}u. \tag{4}$$

Participants should now apply the Leibniz integral rule, the integrand being $f(\omega,u) = e^{-u^2}\cos(2\omega u)$. So we can write the differentiated integral as,

$$I'(\omega) = \int_0^\infty \frac{\mathrm{d}f(\omega,u)}{\mathrm{d}\omega}\mathrm{d}u, \tag{5}$$

and then compute the differential in the new integrand,

$$\frac{\mathrm{d}f(\omega,u)}{\mathrm{d}\omega} = \frac{\mathrm{d}}{\mathrm{d}\omega}e^{-u^2}\cos(2\omega u) = e^{-u^2}(-2u)\cdot\sin(2\omega u). \tag{6}$$

Take this and substitute into Equation 5, yielding

$$I'(\omega) = \frac{1}{a}\int_0^\infty e^{-u^2}(-2u)\cdot\sin(2\omega u)\,\mathrm{d}u. \tag{7}$$

We can now apply integration by parts $\int_o^\infty UV' = UV\Big|_0^\infty - \int_o^\infty U'V$.

Participants may wonder why 2 was not included in the definition of ω, this is because rather neatly the derivative with respect to u of e^{-u^2} produces $-2ue^{-u^2}$ which can be seen in Equation 6.

We let $U = \frac{1}{a} sin\left(2\omega u\right)$ and $V' = -2ue^{-u^2}$, carrying out the relevant differentiation and integration of these with respect to u give us the required inputs of $U' = \frac{2\omega}{a} cos\left(2\omega u\right)$ and $V = e^{-u^2}$.

$$I'\left(\omega\right) = \left(\frac{1}{a} sin\left(2\omega u\right)\right)\left(e^{-u^2}\right)\Big|_0^{\infty} - \int_o^{\infty} \left(e^{-u^2}\right)\left(\frac{2\omega}{a} cos\left(2\omega u\right)\right) \mathrm{d}u, \tag{8}$$

it is clear that applying the limits to the first term will yield zero, so we are left with just one integral for $I'\left(\omega\right)$ that is very similar to $I\left(\omega\right)$.

$$I'\left(\omega\right) = -\int_o^{\infty} \left(e^{-u^2}\right)\left(\frac{2\omega}{a} cos\left(2\omega u\right)\right) \mathrm{d}u \tag{9}$$

$$I\left(\omega\right) = \frac{1}{a}\int_o^{\infty} e^{-u^2} cos\left(2\omega u\right) \mathrm{d}u \tag{10}$$

We now have a very simple differential equation of the form $I'\left(\omega\right) = -2\omega I\left(\omega\right)$, we now know computing the differential of the integral must yield -2ω which means $I\left(\omega\right)$ must have the form of $C \cdot e^{-\omega^2}$, where C is a constant which we can find by applying boundary conditions.

Checking the boundary condition of when $\omega = 0$,

$$I\left(0\right) = \frac{1}{a}\int_0^{\infty} e^{-u^2} \cos\left(0\right) \mathrm{d}u = \frac{1}{a}\frac{\sqrt{\pi}}{2} = C. \tag{11}$$

Therefore, we can write the original integral in terms of u and ω as,

$$I\left(\omega\right) = \frac{1}{a}\int_0^{\infty} e^{-u^2} \cos\left(2\omega u\right) \mathrm{d}u = \frac{1}{a}\frac{\sqrt{\pi}}{2} \cdot e^{-\omega^2}, \tag{12}$$

note that the final result only depends on ω which we defined originally as $\omega = \frac{b}{a}$. This proves that the original integral,

$$I\left(a, b\right) = \int_0^{\infty} e^{-a^2x^2} \cos\left(2bx\right) \mathrm{d}x = \frac{\sqrt{\pi}}{2a} e^{-\frac{b^2}{a^2}}. \tag{13}$$

Award <u>one mark</u> for complete proof only.

Q8

IOP Questions

Q1): Consider Figure 1 (a) which shows a uniform two dimensional domino of height $2l$ and width $2d$ with $d < l$. The domino is placed such that the bottom right bottom corner is at the origin and it follows that the centre of mass of the domino is at the position $(-d, -l)$.

For $d < l$ the domino has lower energy when lying on its side, so that the centre of mass is at the position (l, d), see Figure 1 (b). Why doesn't the domino achieve its minimum energy by spontaneously falling on its side?

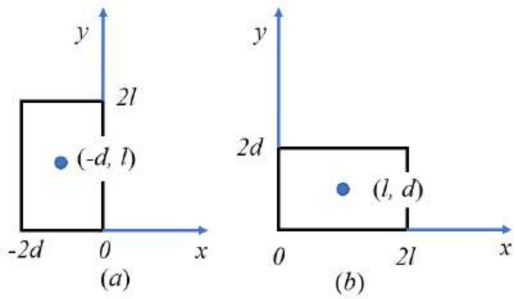

Figure 1: Domino in (a) high energy and (b) low energy state.

Answer): Consider Figure 2 which shows the domino rotated by the angle θ about the right hand lower corner with the angle θ in the range $0 \leq \theta \leq \pi/2$: Implementing

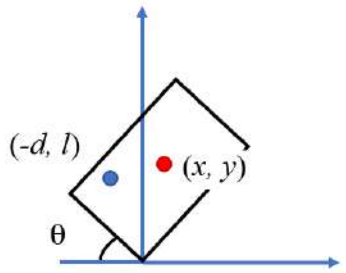

Figure 2: The centre of mass moves from $(-d, l)$ to (x, y) by the active rotation $\mathcal{M}(\theta)$.

the active rotation $\mathcal{M}(\theta)$ gives

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathcal{M}(\theta) \begin{pmatrix} -d \\ l \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} -d \\ l \end{pmatrix} \quad (1)$$

i.e.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -d\cos\theta + l\sin\theta \\ d\sin\theta + l\cos\theta \end{pmatrix}. \tag{2}$$

A quick check by putting $\theta = \pi/2$ gives $(x, y) = (l, d)$ as expected, giving us confidence that the supposed transformation is indeed correct.
Let us consider the result for y:

$$y = d\sin\theta + l\cos\theta. \tag{3}$$

Using the well known small angle expansions for $\sin\theta$ and $\cos\theta$, namely

$$\sin\theta \approx \theta, \cos\theta \approx 1 - \theta^2/2,$$

yields

$$y \approx l + d\theta - l\theta^2/2. \tag{4}$$

It follows that the centre of mass of the domino initially *rises* before getting smaller (taking $d < l$), since for θ small enough the term $d\theta$ dominates over the term $l\theta^2/2$. This can be seen more clearly by writing y in the form

$$y = \sqrt{l^2 + d^2}\sin(\theta + \phi), \tag{5}$$

where the angle $\phi = \sin^{-1} l/\sqrt{l^2 + d^2}$ and we have

$$\phi = \tan^{-1}\frac{l}{d} = \begin{cases} 0 < \phi \leq \frac{\pi}{4} & \text{if } d \geq l \\ \frac{\pi}{4} < \phi < \frac{\pi}{2} & \text{if } l > d. \end{cases}$$

Thus, the reason the domino cannot spontaneously fall over is because there is a potential barrier that must be overcome before the minimum energy can be achieved. We consider in some detail the expression for y:

$$y = l\cos\theta + d\sin\theta,$$
$$\frac{dy}{d\theta} = -l\sin\theta + d\cos\theta,$$
$$\frac{d^2y}{d\theta^2} = -(l\cos\theta + d\sin\theta).$$

Setting $dy/d\theta = 0$ gives $\tan\theta_m = d/l$, *i.e.* $\theta_m = \tan^{-1} d/l$. We note that at this angle $\theta_m + \phi = \frac{\pi}{2}$ giving $\sin(\theta_m + \phi) = 1, y_m = \sqrt{l^2 + d^2}$ and since $d^2y/d\theta^2$ is clearly negative, it follows that this is a maximum. Figure 3 shows the variation of the height of the centre of mass with the angle θ for a value d such that $d < l$.
From the figure we see the centre of mass is a height l above the ground when $\theta = 0$,

Q8

and increases as θ increases until it reaches the maximum height of $\sqrt{l^2 + d^2}$ when $\theta = \theta_m = \tan^{-1} dl$. As θ increases beyond this value the centre of mass falls to the height d. In the figure there are two values of θ where the height of the centre of mass is equal to l, at $\theta = 0$ and at the value θ^*. The angle θ^* is given as

$$\theta^* = \pi - 2\phi.$$

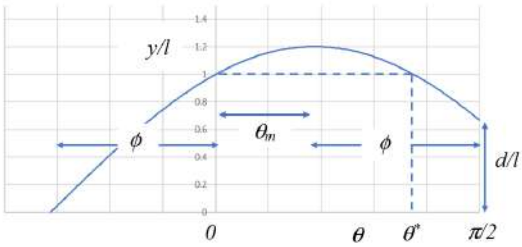

Figure 3: The variation of the height of the centre of mass with θ.

Q2): Figure 4 shows a simple pendulum, where a bob of mass m is suspended by a light, inextensible and taut rod. The bob is displaced by an angle θ and moves under the influence of gravity with the acceleration due to gravity is g. The equation of

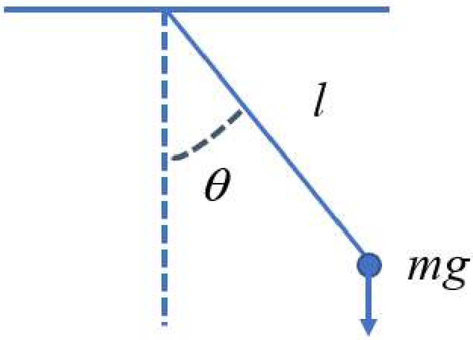

Figure 4: The simple pendulum.

motion when the bob is displaced by the angle θ is easily derived and found to be

$$\frac{d^2\theta}{dt^2} + \frac{g}{l} \sin\theta = 0. \tag{6}$$

For small angles the approximation $\sin\theta \approx \theta$ is valid and the equation of motion becomes

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0, \tag{7}$$

and the solution for the ensuing motion when the initial conditions are $\theta() = theta + 0, d\theta/dt(0) = 0$, $i/e/$ the bob starts from rest from the angle θ_0 is

$$\theta(t) = \theta_0 \cos\sqrt{\frac{g}{l}} t\theta = 0. \tag{8}$$

One implication of this is that the period of the motion, T, is given as $T = 2\pi\sqrt{l/g}$ and is independent of the amplitude (as long as the amplitude is small enough such that the approximation of $\sin\theta \approx \theta$ remains valid during the motion).
Present a simple argument based on continuity that suggests that the period increases as the initial amplitude increases such that the approximation of small angle is not justified.
Answer): Figure 5 shows the pendulum where the initial angle of deflection is $\theta_0 = \pi$. In this position the bob is in (unstable) equilibrium and, in the absence of any other

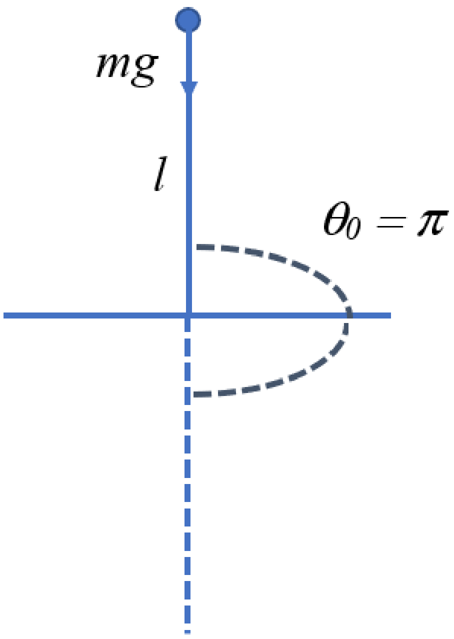

Figure 5: The case $\theta_0 = \pi$.

forces, will remain in this position for ever. Denoting by $T(\theta_0)$ the period of the motion that depends on the initial amplitude we have $T(\theta_0) \approx 2\pi\sqrt{l/g}$ independent of the initial amplitude for small θ_0 increasing to infinity for an initial amplitude $\theta_0 = \pi$. Assuming there is a continuous variation of the period with initial amplitude we expect the amplitude to increase with θ_0.

Q9

PLANCKS 2020 Finals
Buried Room Problem Solutions

This questions is a very open-ended question, it is designed to see what assumptions the participants make and what conclusions they can form from the information given and their own assumptions. Therefore, there is no exact mark scheme for this problem, instead we will award marks for reasonable assumptions, taking a logical argument from these assumptions using fundamental physics to reach their answer to the problem.

Assumptions and Conclusions regarding the Person

Award one mark for reasoning along these lines.

- The question asks if you will be standing on one of the walls or floating. The question does not refer to your body or corpse, so participants should make the assumption that they can survive the immense pressure and temperature at the centre of the Earth.
- If carrying out any calculations participants should make the assumption that the person in the room is a point mass and give a reasonable value for their mass 100 kg.
- Assuming the person is not Nicolas Cage- warning do not unleash The Cage on this problem then anything is possible.

Assumptions and Conclusions regarding the Earth

- An obvious simplification is to make the Earth be completely spherical and have a homogeneous density. Having such a system would result in having equal attraction from all directions if you were floating in the direct centre of the Earth (this is just logical no need to state any equations for this).
  Award one to two marks for reasoning along these lines.

  One obvious approach to the problem leading on from the assumptions above can be made just using a simple diagram and coherent logic. Figure 1, shows what the influence at the surface would be all round the globe is the same. Then with some argument following the lines of how the gravity of a solid sphere decreases linearly from the surface to the centre (sphere inside sphere proof) then they should be able to explain why at the centre the force experienced is zero.

  Should they provide a nice diagram and an explanation along these lines then award one to two marks in addition to those awarded for the assumptions above.
- If participants have assumed the Earth is spherical-like and has a density which varies with position, they must state how having different densities will affect the gravitational force on the person in the room e.g. if a person was started in the middle of the Earth and on one side there was an volume of higher density than the other side, the mass on that side would be greater and the person would be pulled towards that wall.
  Award one to two marks for reasoning along these lines.

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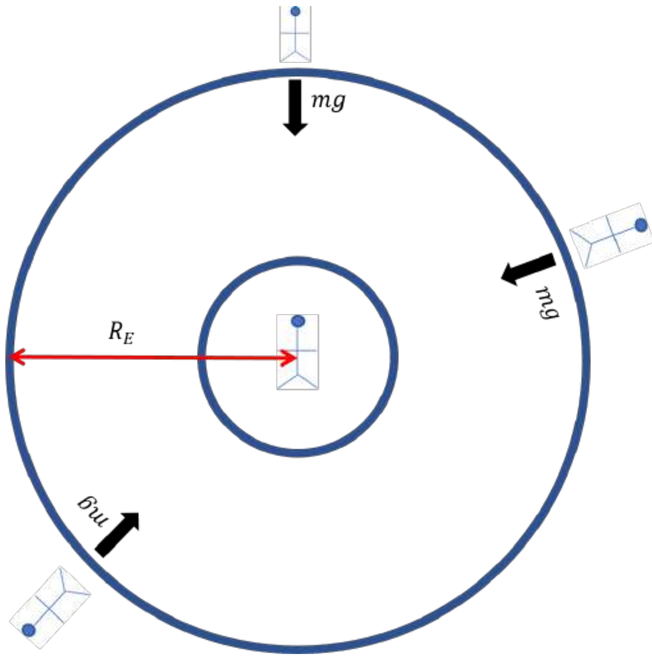

Figure 1: Diagram of the Earth under the assumption that is is a sphere and has uniform density and therefore uniform gravity.

Assumptions and Conclusions regarding the Solar System

- The obvious simplification is to say that the Moon and the Sun have negligible influence on a person of 80 kg being at the centre of the Earth due to the extreme distances they are at. This requires some backup using Newton's law of gravitation and stating how the force decays with the inverse square law.
  Award one to two marks for reasoning along these lines.

- If they assume that the Sun or Moon still has significant influence on the person in the room participants must show that this is the case either via "back of the envelope calculations" (see below for example) or by logical reasoning e.g. the Moon influences the tides (although that is a real stretch).
  Award one to two marks for reasoning along these lines.

  A participant could show that the Sun still had a role to play in the scenario by looking at Newton's law of gravitation. We have at some point concluded the person side the shell should not experience a force but are their other things we could extend to have great influence on them e.g. another person in the room and would this compare to the effect the Sun has on the person if any?

  Taking $M_{Sun} \approx 2 \cdot 10^{30}$ Kg and $1Au \approx 1.5 \cdot 10^{11}$ m we can make some interesting comparisons of influence.

  For another person in the room at 1 m distance,

$$G\frac{100 \cdot 100}{1^2} \sim 10^4 G \tag{1}$$

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and for the Sun and a person in the room,

$$G\frac{2 \cdot 100 \cdot 10^3}{1.5^2 \cdot 10^{22}} \sim 10^{10}G \tag{2}$$

clearly the Sun has more influence on a person in the room than another person in the room and participants work if they have mentioned the Sun or other bodies in the Solar System should reflect this.

Assumptions and Conclusions regarding the Room

Award one mark for reasoning along these lines.

- Are you alone in the room or are there other people? What would be the effect of the others in the room (see calculations above for starters), if they assumed that nothing else except object in the room have an effect can they give some interesting ideas as to what might happen e.g. how long would it take two people to be attracted to each other in the room?
- What are the walls made out of? Obviously, something ridiculously strong and heat resistant but what impact does this have on the problem? Do they assume the walls are magic and have no density, what are the consequences if they do?
- Does the shape of the room matter? Following all the other arguments in this solution we like spheres and for most of the maths we have assumed the room as a sphere, but would the box shape actually change the force inside the room because the mass around the person from the Earth is no longer spherically symmetric? It would certainly make any integrals the participant tries to do more difficult, but in the end comparing the size of the Earth to the room probably not.

Mathematical Approaches to the Problem

Participants may want to take a mathematical approach to this problem, here two abridged methods are given to show that the gravitational field strength inside room will be zero.

Thin Shell

The first case is for a thin shell and you may award a maximum of three marks with reasonable assumptions.

Showing that for a thin spherical shell (approximating the room in the middle to be a sphere and the rest of the Earth to be a thin sphere). In this case they can show that the gravitational field strength inside the shell will be zero. The full method can be found in Physics for Scientist and Engineers by Tipler and Mosca.

Take the field at a point P which is due to the element $\mathrm{d}m$ of an infinitesimally thick ring, P is at a distance x to the centre, the distance of the element to P is s and this makes an angle α with the horizontal.

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The field at point P has a magnitude,

$$\mathrm{d}g = G\frac{\mathrm{d}m}{s^2} \tag{3}$$

using symmetry we can see all the perpendicular components contributing will cancel out and the x component of the field will be given by,

$$\mathrm{d}g_x = -\mathrm{d}g\cos\alpha = -G\frac{\mathrm{d}m}{s^2}\cos\alpha. \tag{4}$$

Integrate over the whole ring giving the x component of the field strength to be,

$$g_x = -G\frac{m}{s^2}\cos\alpha, \tag{5}$$

where m is the total mass of the ring.

With the same argument, we can apply this to a spherical shell of mass M and radius R yielding,

$$\mathrm{d}g_r = -G\frac{\mathrm{d}M}{s^2}\cos\alpha \tag{6}$$

where $\mathrm{d}M$ is the mass element of the shell and is given by $\mathrm{d}M = \frac{M}{A}\mathrm{d}A = \frac{M}{4\pi R^2}2\pi R^2\sin\theta\mathrm{d}\theta$ where A is the area of the shell. Giving the following expression for the infinitesimal field,

$$\mathrm{d}g_r = -G\frac{M\sin\theta\mathrm{d}\theta}{2s^2}\cos\alpha \tag{7}$$

with a bit of trigonometry a relationship between s, θ, α, r and R can be found which gives us our integratable result of,

$$\mathrm{d}g_r = \frac{-GM}{4r^2R}\left(1+\frac{r^2-R^2}{s^2}\right)\mathrm{d}s. \tag{8}$$

For this scenario we are interested in the point P being inside the sphere so the field we are interested in varies from $R-r$ to $R+r$. Therefore, our integrated expression for the field is,

$$g_r = \frac{-GM}{4r^2R}\left(s-\frac{(r-R)(r+R)}{s}\right)\Bigg|_{R-r}^{R+r} \tag{9}$$

clearly when putting in the limits this yields zero. So, what about force? Well if the person was of mass m and we treated them as our point P throughout then to get the force use Newton's second law $F = mg$ and it is clear the net force inside the sphere acting on the person is zero.

Thick Shell

The second case is *slightly more accurate* and you may award a maximum of five marks with reasonable assumptions.

Showing that for a thick shell of finite thickness (approximating the room to be a sphere and the thick shell of the Earth to have a thickness of $R_{Earth} - R_{Room}$). In this case they can show that the gravitational field strength inside the shell will be zero.

To do this one we look at gravitational potential because that's what all the literature does and Watford is too far away for me to grab my 1^{st} year notes on the problem.

We shall look at the gravitational potential Φ of a shell with a finite thickness. The shell has mass M, density ρ, an inner radius of a and and outer radius of b. To compute this we look at a point P which is at a distance R from the centre of the shell and a distance r from the source of the potential.

The gravitational potential given in terms of the density,

$$\Phi = -G \int \frac{\rho\left(r'\right) \mathrm{d}V'}{r} \tag{10}$$

the infinitesimal volume can be written as $\mathrm{d}V' = r'^2 \sin\theta \mathrm{d}r' \mathrm{d}\theta \mathrm{d}\phi$. The $\mathrm{d}\phi$ term yields a 2π in the integration and we are assuming a constant density (if participants wish to have a density which varies with radius, good luck to them) this gives us a formula for the potential of,

$$\Phi. = -2\pi\rho G \int_b^a r'^2 \mathrm{d}r' \int_0^\pi \frac{\sin\theta \mathrm{d}\theta}{r}, \tag{11}$$

using some trigonometry we can substitute $\frac{\mathrm{d}r}{r'R}$ rid of the θ term and out in terms of r the distance from the source to the point P.

The potential is now,

$$\Phi = \frac{-2\pi\rho G}{R} \int_b^a r' \mathrm{d}r' \int_{r_{min}}^{r_{max}} \mathrm{d}r, \tag{12}$$

we are interested in points inside the shell i.e $R < a$,

$$\Phi = \frac{-2\pi\rho G}{R} \int_b^a r' \mathrm{d}r' \int_{r'-R}^{r'+R} \mathrm{d}r. \tag{13}$$

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Evaluating these integrals gives us a constant value of the potential everywhere inside the shell,

$$\Phi = -2\pi\rho G\left(b^2 - a^2\right) = constant. \tag{14}$$

The gravitational field strength which relates the mass of the person m to the force they experience is given by $g = -\nabla\Phi$ and when $R < a$ the potential is a constant therefore the field strength is zero and so if the force experienced by the person inside the "hollow Earth".

In either of these cases showing that the person in the box does not experience a force due to the Earth (with all the prerequisites that come with that statement), they can say that other things will influence the person in the box i.e. the other ideas discussed in this "solution" to the problem.

A Final Note

If the participants make reference at any point in their answering to the 2008 film 'Journey to the center of the Earth' starring Brendan Fraser, the answer or argument they are currently trying to present should be taken as Universal canon and you can award a maximum of six marks.

Summary

Here is a summary on the the awarding of marks for this problem. However, the final decision on such an open ended question on where marks shall be awarded rests with the marker. The marker should consistent with their allocation of marks and may find the discussion above useful. A reminder that the maximum marks that can be awarded for this question is ten marks.

(i) A discussion on the gravitational field and force in side the Earth will merit between one and six marks. Concluding that the person at the centre of the Earth experiences zero force with some reasonable assumptions (one to two marks), with a diagram (extra one mark) and with a full set of assumptions and a reasonable mathematical approach (five to six marks).

(ii) A discussion on other objects in the Solar System and their influence or lack of influence will merit between one and two marks.

(iii) A discussion on the room and its properties can merit up to one mark.

(iv) A discussion on the person or number of people can merit up to one mark.

(v) A discussion on other things that could happen to the system that have not been discussed here can merit an additional one to two marks if they are reasonable and have sufficient logic behind the.