DOPPLERS

Problem 5

Gravitational decoherence

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Background Realistic physical quantum systems are usually not isolated but in continuous interaction with some environment which can lead to an exchange of energy and information. If one is only interested in the behaviour of a specific part ("core system") of the entire system, one considers this part as so-called open quantum system and neglects the detailed evolution of the environment; only its effective influence on the core system is considered. This treatment leads to effects like dissipation (loss of energy) or decoherence (loss of information) in the core system. In this problem you will work out a toy model system for decoherence induced by gravity on some matter system which we consider as core system. Inspired by field theory, we will model the matter system under consideration as a harmonic oscillator and the environment as a bath of harmonic oscillators modelling gravitational waves/gravitons that transfer the gravitational interaction in our model here. The structure of the coupling term in the Lagrangian for this system is then determined by general relativity:

$$L = \frac{1}{2}M\dot{x}^2 - \frac{1}{2}M\Omega^2x^2 + \sum_i \left(\frac{1}{2}m\dot{q}_i^2 - \frac{1}{2}m\omega_i^2 q_i^2\right) - \lambda\left(\frac{1}{2}M\dot{x}^2 + \frac{1}{2}M\Omega^2x^2\right)\sum_i q_i\,. \tag{5.1}$$

The harmonic oscillator modelling the matter part has constant mass M, positive frequency Ω and position variable x, the oscillators for the environment, labelled with integers i, have constant and equal masses m, positive frequencies ω_i and position variables q_i. The coupling, moderated by the coupling parameter λ, links the energy of the matter system with the configuration (=position) variable of the environment. Note that the entire Lagrangian has no explicit time dependence.

a) [*0.5 points*] Use the Euler-Lagrange equations to show that without coupling (that is for $\lambda = 0$) one indeed obtains the uncoupled equations of motion for a harmonic oscillator in the core system and one for each i in the environment.

b) [*1 point*] The size of the coupling parameter λ describes the strength of the gravitational interaction between the matter system and the environment. While for arbitrary λ the treatment and physical justification of such a toy model might be difficult because one would in general need a full theory of quantum gravity to describe such a system in an appropriate way, we focus on weak interactions (i.e. small λ), where one can work with perturbation theory and also follow a Fock quantisation in the full field theory case. Perform the Legendre transformation of the above Lagrangian and use the assumption of weak coupling to arrive at the following Hamiltonian:

$$H = \left(\frac{p^2}{2M} + \frac{1}{2}M\Omega^2x^2\right)\left(1 + \lambda\sum_i q_i\right) + \sum_i\left(\frac{p_i^2}{2m} + \frac{1}{2}m\omega_i^2q_i^2\right)\,, \tag{5.2}$$

where p denotes the canonically conjugated momentum to x and p_i to q_i.

c) [*0.5 points*] In a next step, we want to quantise this system on a Hilbert space $\mathcal{H}_{SE} = \mathcal{H}_S \otimes \mathcal{H}_E$ in the standard fashion by quantising the single harmonic oscillators, where the one for the matter system lives on $\mathscr{H}_S$ and the ones of the environment on $\mathscr{H}_E$. Use Ladder operators a (for the core system) and b_i (for the environment),

$$a := \sqrt{\frac{M\Omega}{2\hbar}}\left(x + \frac{i}{M\Omega}p\right) \qquad [a, a^\dagger] = 1, \qquad [a, a] = [a^\dagger, a^\dagger] = 0 \tag{5.3}$$

$$b_i := \sqrt{\frac{m\omega_i}{2\hbar}}\left(q_i + \frac{i}{m\omega_i}p_i\right) \qquad [b_i, b_j^\dagger] = \delta_{ij} \qquad [b_i, b_j] = [b_i^\dagger, b_j^\dagger] = 0\,, \tag{5.4}$$

to show that the quantised Hamiltonian can be written as

$$H = H_S \otimes \mathbb{1}_E + \mathbb{1}_S \otimes H_E + \lambda H_{SE} \tag{5.5}$$

with

$$H_S = \hbar\Omega\left(a^\dagger a + \frac{1}{2}\right) \qquad H_E = \sum_i \hbar\omega_i\left(b_i^\dagger b_i + \frac{1}{2}\right) \tag{5.6}$$

$$H_{SE} = \hbar\Omega\left(a^\dagger a + \frac{1}{2}\right) \otimes \sum_i \sqrt{\frac{\hbar}{2m\omega_i}}\left(b_i + b_i^\dagger\right)\,. \tag{5.7}$$

Note that $a^{(\dagger)}$ and $b_i^{(\dagger)}$ live on different Hilbert spaces and hence commute.

d) [*0.5 points*] An open quantum system is usually described in the density operator formalism. Given a Hilbert space $\mathscr{H}$ with normalised states $\{|\psi_A\rangle\}$, a density operator can be written down as

$$\rho = \sum_A p_A \left|\psi_A\right\rangle \left\langle\psi_A\right|\,, \tag{5.8}$$

where the p_A are non-negative coefficients and the trace of the density matrix $\mathrm{Tr}(\rho) = 1$. Working in the Schrödinger picture, i.e. with constant p_A, show that the Schrödinger equation for the wave function implies the Liouville-von Neumann equation that describes time evolution of the density operator:

$$\frac{d}{dt}\rho(t) = -\frac{i}{\hbar}[H, \rho(t)]\,, \tag{5.9}$$

where you can assume the Hamiltonian to be self-adjoint (as is the one in this problem).

e) [*1.5 points*] To describe time evolution, one uses besides the Schrödinger picture (where the time dependence is in the wave function) and the Heisenberg picture (where the time dependence is in the operators) also the so-called interaction picture, where both wave function $|\psi\rangle$ and operators A are time-dependent and evolve in the following way. In our notation of the open quantum system introduced above, the transformation of these quantities from Schrödinger to interaction picture is

$$|\psi(t)\rangle \longrightarrow \widetilde{|\psi(t)\rangle} = U_0^\dagger(t,0)\,|\psi(t)\rangle \tag{5.10}$$

$$A \longrightarrow \widetilde{A} = U_0^\dagger(t,0) A U_0(t,0) =: \widetilde{A}(t)\,, \tag{5.11}$$

where $|\psi(t)\rangle = U(t,0)\,|\psi(0)\rangle$ denotes the state in Schrödinger picture, U is the time evolution operator of the entire system, i.e. the solution of $i\hbar\frac{d}{dt}U(t,0) = HU(t,0)$, and U_0 is the time evolution operator of the $\lambda = 0$ system. Show that in the interaction picture the Liouville-von Neumann equation of the entire system reads

$$\frac{d}{dt}\widetilde{\rho}(t) = -\frac{i}{\hbar}\lambda\left[\widetilde{H}_{SE}(t), \widetilde{\rho}(t)\right] \tag{5.12}$$

with $\widetilde{\rho}(t) := U_0^\dagger(t,0)\rho(t)U_0(t,0)$ and determine $\widetilde{H}_{SE}(t)$ explicitly.
Hint: Use the fact that $e^A B e^{-A} = \sum_{m=0}^{\infty} \frac{1}{m!}[A,B]_{(m)}$, where $[A,B]_{(m)}$ denotes the iterated commutator defined recursively by $[A,B]_{(m)} := [A,[A,B]_{(m-1)}]$ and $[A,B]_{(0)} = B$.

f) [*0.5 points*] Show that the evolution equation in (5.12) can be rewritten as

$$\frac{d}{dt}\widetilde{\rho}(t) = -\frac{i}{\hbar}\lambda[\widetilde{H}_{SE},\rho(0)] - \frac{\lambda^2}{\hbar^2}\int_0^t dt'[\widetilde{H}_{SE}(t),[\widetilde{H}_{SE}(t'),\widetilde{\rho}(t')]]\,. \tag{5.13}$$

g) [*2 points*] As in the end we are only interested in the behaviour of the core system under the effective influence of the environment, we can trace out the environment in the above Liouville-von Neumann equation. This partial trace over the environmental Hilbert space $\mathcal{H}_E$ yields a so-called master equation which is a time evolution equation only for $\widetilde{\rho}_S(t) := \mathrm{Tr}_E\{\widetilde{\rho}(t)\}$,

$$\frac{d}{dt}\widetilde{\rho}_S(t) = -\frac{i}{\hbar}\lambda \mathrm{Tr}_E\left\{[\widetilde{H}_{SE},\rho(0)]\right\} - \frac{\lambda^2}{\hbar^2}\int_0^t dt'\ \mathrm{Tr}_E\left\{[\widetilde{H}_{SE}(t),[\widetilde{H}_{SE}(t'),\widetilde{\rho}(t')]]\right\}\,. \tag{5.14}$$

To evaluate this further, we invoke a set of assumptions:

- In the last term in the master equation we can replace $\widetilde{\rho}(t') \longrightarrow \widetilde{\rho}_S(t)\otimes\widetilde{\rho}_E(t)$. This is the Born and the first Markov approximation and can be used here because we work with a small coupling constant λ.
- At time $t=0$ the core system and environment were in a product state, i.e. $\rho(0) = \rho_S(0)\otimes\rho_E(0)$.
- The environment is in a steady thermal state $\widetilde{\rho}_E(t) = \rho_E(0) = \rho_E = \frac{1}{Z}e^{-\beta H_E}$ with positive temperature parameter $\Theta := \frac{1}{\beta}$.

Show that the master equation then simplifies to

$$\frac{d}{dt}\widetilde{\rho}_S(t) = -\frac{\lambda^2}{\hbar^2}\int_0^t dt'\ \mathrm{Tr}_E\left\{[\widetilde{H}_{SE}(t),[\widetilde{H}_{SE}(t'),\widetilde{\rho}_S(t)\otimes\rho_E]]\right\} \tag{5.15}$$

and determine the normalisation factor Z in ρ_E as well as ρ_E from the requirement that $\mathrm{Tr}_E\{\rho_E\} = 1$.
Hint: Use the occupation number basis to evaluate the partial trace and use the limit of the geometric series: $\sum_{m=0}^{\infty} ax^m = \frac{a}{1-x}$ for $|x|<1$.

h) [*2 points*] Explicitly compute the double commutator in the master equation and show that

$$\frac{d}{dt}\widetilde{\rho}_S(t) = -\frac{\lambda^2}{\hbar^2}\Big[-2f(t)\Big(H_S\widetilde{\rho}_S(t)H_S - \frac{1}{2}\{H_S^2,\widetilde{\rho}_S(t)\}\Big) + g(t)(H_S^2\widetilde{\rho}_S(t) - H_S\widetilde{\rho}_S(t)H_S) + g^*(t)(\widetilde{\rho}_S(t)H_S^2 - H_S\widetilde{\rho}_S(t)H_S)\Big]\,, \tag{5.16}$$

where a star denotes complex conjugation, $\{X,Y\} := XY+YX$ the anti-commutator and the functions $f(t)$ and $g(t)$ are given by

$$f(t) := \int_0^t dt' \sum_i \frac{\hbar}{2m\omega_i}2N(\omega_i)\cos[\omega_i(t-t')] \tag{5.17}$$

$$g(t) := \int_0^t dt' \sum_i \frac{\hbar}{2m\omega_i}e^{-i\omega_i(t-t')} \tag{5.18}$$

with the Bose-Einstein distribution $N(\omega) := \frac{1}{e^{\beta\hbar\omega_i - 1}}$.
Hint: Use the occupation number basis to evaluate the partial trace.

i) [*0.5 points*] With additional approximations and calculation techniques, it is possible to evaluate $f(t)$ and $g(t)$ further. Here we won't consider the details; we just assume this has been carried out resulting in $F\Theta := \frac{\lambda^2}{\hbar^2} f(t)$ being time independent, real and linear in the temperature parameter $\Theta = \frac{1}{\beta}$ and $iG := \frac{\lambda^2}{\hbar^2} g(t)$ being also time independent and purely imaginary. Transform the master equation back into the Schrödinger picture and show that a solution to it in the occupation number basis, i.e. in the basis $\rho_S(t) = \sum_{m,n} \rho(m,n,t) |m\rangle \langle n|$, is then given by

$$\rho_S(m,n,t) = e^{-\frac{i}{\hbar}(E_m - E_n)t - \Theta F (E_m - E_n)^2 t - iG(E_m^2 - E_n^2)t} \rho_S(m,n,0)\,, \tag{5.19}$$

where $H_S |m\rangle = E_m |m\rangle$.

j) [*1 point*] Argue why the term containing G corresponds to an energy renormalisation due to the presence of the environment. Show that the master equation leaves the populations in the density matrix in occupation basis (i.e. the $\rho(m,m,t)$) unaffected.
The term containing F leads to decoherence due to the presence of the environment (gravitationally induced decoherence). Any part of the density matrix except the populations are damped and therefore will decrease with time, making the density matrix in occupation number basis approach a diagonal matrix with no interference probabilities and hence becoming a classical system. How does a change by a factor of two of the temperature parameter Θ, the coupling parameter λ and the occupation number difference $m - n$ affect the decoherence?