Problem 2

## Superselection

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**Background** If $\psi_1$, $\psi_2$ are wave functions of two quantum states (i.e. elements of a Hilbert space with scalar product $\langle\cdot|\cdot\rangle$), any superposition

$$\psi = c_1\psi_1 + c_2\psi_2,\ c_1, c_2 \in \mathbb{C}, \tag{2.1}$$

is a wave function of a quantum state, too. A characteristic feature of quantum mechanics is that the relative phase between $\psi_1$ and $\psi_2$ is in general observable: Multiplying a wave function by a complex number of modulus 1 does not change the corresponding physical state, but

$$\psi(\alpha) = \mathrm{e}^{\mathrm{i}\alpha} c_1\psi_1 + c_2\psi_2,\ \alpha \in [0, 2\pi), \tag{2.2}$$

is in general a wave function of a different quantum state than $\psi$. However, if $\psi_1$ and $\psi_2$ are separated by a **superselection rule**, that is,

$$\langle\psi_1|A\psi_2\rangle = 0 \text{ for all observables } A, \tag{2.3}$$

then the relative phase is not observable (see part a)). In this problem, we study superselection rules for a quantum particle confined to a closed loop.

**a)** [*1 point*] Verify that the relative phase between $\psi_1$ and $\psi_2$ is not observable (i.e. $\langle\psi(\alpha)|A\psi(\alpha)\rangle = \langle\psi|A\psi\rangle$ for all observables $A$) if $\psi_1$ and $\psi_2$ are separated by a superselection rule.

We analyse a quantum particle confined to a closed loop with circumference $L$. We model the state space of such a particle by $L$-periodic wave functions $\psi$ with phase shift $\theta \in [0, 2\pi)$ (i.e. $\psi(x + L) = \mathrm{e}^{\mathrm{i}\theta}\psi(x)$), and we denote the Hilbert space of all such functions by $\mathcal{H}_\theta$. The scalar product in $\mathcal{H}_\theta$ is defined as follows:

$$\langle\varphi|\psi\rangle_{\mathcal{H}_\theta} = \int_0^L \overline{\varphi(x)}\psi(x)\,\mathrm{d}x. \tag{2.4}$$

**b)** [*1 point*] Consider the momentum operator $P_\theta = -\mathrm{i}\partial_x$ acting on $\mathcal{H}_\theta$. Show that $P_\theta$ is hermitian (i.e. $\langle\varphi|P_\theta\psi\rangle_{\mathcal{H}_\theta} = \langle P_\theta\varphi|\psi\rangle_{\mathcal{H}_\theta}$ for wave functions $\varphi, \psi$ that are $L$-periodic with phase shift $\theta$).

**c)** [*1 point*] Compute the eigenvalues and the normalised eigenfunctions of $P_\theta$.

**d)** [*2 points*] Consider the (normalised) wave function $\psi$ defined as the unique extension of

$$\psi(x) = \sqrt{30/L^5}\,x(x - L),\ x \in [0, L), \tag{2.5}$$

to an $L$-periodic wave function with phase shift $\theta \in [0, 2\pi)$. Compute the probability that the momentum $(2\pi k + \theta)/L$, $k \in \mathbb{Z}$, is measured in the state $\psi$. Verify that the probability depends on $\theta$ but not on $L$. *Hint:* $\int_0^1 \mathrm{e}^{-\mathrm{i}2ax}x(x-1)\,\mathrm{d}x = \frac{\mathrm{e}^{-\mathrm{i}a}}{2a^3}(a\cos(a) - \sin(a))$, $a \in \mathbb{R}$.

**e)** [*2 points*] Prove that $\theta$ is a superselection rule (i.e. $L$-periodic wave functions with different phase shifts are separated by a superselection rule).

The space $\mathcal{H}_\theta$ is a **superselection sector** because $\theta$ is a superselection rule. The (continuous) direct sum of all superselection sectors,

$$\mathcal{H} = \frac{1}{2\pi} \int_{[0,2\pi)}^{\oplus} \mathcal{H}_\theta \, \mathrm{d}\theta, \tag{2.6}$$

is the physical Hilbert space.[1] Accordingly, the (self-adjoint) physical momentum operator is the direct sum of all $P_\theta$:

$$P = \frac{1}{2\pi} \int_{[0,2\pi)}^{\oplus} P_\theta \, \mathrm{d}\theta. \tag{2.8}$$

**f)** [*1 point*] Apply your result from part c) to determine the (generalised) eigenvalues[2] of the physical momentum operator $P$. *Hint:* If $A$, $B$ are two operators and $\psi$ an eigenvector of $A$, then $\psi \oplus 0$ is an eigenvector of $A \oplus B$.

**g)** [*1 point*] Denote by $L^2(\mathbb{R})$ the Hilbert space of all square-integrable wave functions on $\mathbb{R}$, and consider the following operator that maps $L^2(\mathbb{R})$ to $\mathcal{H}$:

$$(U\varphi)_\theta(x) = \sum_{k=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}k\theta} \varphi(x + kL), \; \varphi \in L^2(\mathbb{R}). \tag{2.9}$$

Verify that $(U\varphi)_\theta$ is $L$-periodic with phase shift $\theta$ and that $U$ is an isometry (i.e. $\langle U\varphi_1 | U\varphi_2 \rangle_{\mathcal{H}} = \int_{\mathbb{R}} \overline{\varphi_1(x)} \varphi_2(x) \, \mathrm{d}x$ for all square-integrable wave functions $\varphi_1$, $\varphi_2$).

**h)** [*1 point*] In fact, $U$ is a unitary operator and transforms the physical momentum operator $P$ to the momentum operator $\tilde{P} = -\mathrm{i}\partial_x$ on $L^2(\mathbb{R})$ (i.e. $U^\dagger P U = \tilde{P}$, where $U^\dagger$ is the adjoint operator of $U$). Compute the (generalised) eigenvalues of $\tilde{P}$, and compare them with your result from part f). Explain why the generalised eigenvalues of $P$ and $\tilde{P}$ are the same.

[1] An element $\psi = \frac{1}{2\pi} \int_{[0,2\pi)}^{\oplus} \psi_\theta \, \mathrm{d}\theta \in \mathcal{H}$ is a continuous direct sum of elements $\psi_\theta \in \mathcal{H}_\theta$. The scalar product of two elements $\psi_1, \psi_2 \in \mathcal{H}$ is defined as follows:

$$\langle \psi_1 | \psi_2 \rangle_{\mathcal{H}} := \frac{1}{2\pi} \int_0^{2\pi} \langle \psi_{1,\theta} | \psi_{2,\theta} \rangle_{\mathcal{H}_\theta} \, \mathrm{d}\theta = \frac{1}{2\pi} \int_0^{2\pi} \int_0^L \overline{\psi_{1,\theta}(x)} \psi_{2,\theta}(x) \, \mathrm{d}x \, \mathrm{d}\theta. \tag{2.7}$$

[2] Roughly speaking, generalised eigenvalues are eigenvalues whose corresponding eigenvectors are not necessarily normalisable. Example: Every $\lambda \in \mathbb{R}$ is a generalised eigenvalue of the position operator $X = x$ on $L^2(\mathbb{R})$. An eigenvector corresponding to $\lambda$ is the $\delta$-distribution $x \mapsto \delta(x - \lambda)$, which is not a square-integrable function in the usual sense.