# Ricci Flow from Euclidean AQFT

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# Outline of the Talk

1. Motivations and the Ricci Flow
2. Nonlinear Sigma models
3. Euclidean AQFT
4. Non Linear Sigma Models as EAQFT
5. Ricci flow and Renormalization

Based on

- M. Carfora, C. D., N. Drago and P. Rinaldi, arXiv:1809.07652 [math-ph].
- C. D., N. Drago and P. Rinaldi, arXiv:1903.01258 [math-ph].

# The AQFT viewpoint

Algebraic Quantum Field Theory is a two-step approach:

- assign a unital *-algebra $\mathcal{A}$ encoding CCRs, causality, time-slice axiom,....
- assign a state $\omega : \mathcal{A} \to \mathbb{C}$

$$\omega(\mathbb{I}) = 1, \quad \omega(a^* a) \geq 0, \quad \forall a \in \mathcal{A}$$

It is a very effective and rigorous approach to analyze

- QFT on *Lorentzian* curved backgrounds,
- conformal field theories, especially in 2D,
- gauge theories (BRST, BV, ....),
- interacting models form a perturbative viewpoint (pAQFT),
- regularization and renormalization....

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# Ricci Flow

Let $(M, g_0)$ be a **Riemannian manifold** and $I \subseteq \mathbb{R} \ni \beta \mapsto g(\beta) \in \mathrm{Riem(M)}$

$$\frac{\partial g(\beta)}{\partial \beta} = -2Ric[g(\beta)], \quad g(0) = g_0$$

This is the Ricci Flow

It has great relevance:

1. In geometric analysis as a key ingredient in the proof of Thurston geometrization conjecture (Hamilton – 1982 & Perelman – 2003)
2. In theoretical physics as the *renormalization group flow* of nonlinear Sigma models (D. Friedan – Ann. of Phys. **163** (1985) 1057)
3. In averaging cosmology – see M. Carfora's talk SYKM 1.2 – Tue. 17:10

Problem: many derivations but all with some limitations...

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# Nonlinear Sigma Models – basics

Main Ingredients:

- $(\Sigma, \gamma)$ a (connected, oriented) Riemannian manifold $\dim \Sigma = 2$,
- $(M, g)$ a Riemannian manifold with $\dim M = D \geq 2$,
- $\psi \in C^\infty(\Sigma; M)$

Dynamics is encoded in the Lagrangian density

$$\mathcal{L}_H[\gamma, g, \psi] = \mathrm{Tr}_\gamma(\psi^* g)\mu_\gamma \equiv g_{ab}(\psi(x))\gamma^{\alpha\beta} d\psi^a_\alpha d\psi^b_\beta$$

Next step: Expand to 2nd order!

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# NLSM – expansion to 2nd order

Kinematical configurations: $\Gamma(\psi^* TM) \ni \varphi$

- Via the exponential map $\exp_{\psi(x)} : T_{\psi(x)}M \to M$

$$\psi_\nu(x) \doteq \exp_{\psi(x)}(\nu\varphi(x)).$$

The Lagrangian is expanded as:

$$\mathcal{L}_H[\gamma, g, \psi_\nu] = \mathcal{L}_H[\gamma, g, \psi] + \nu g(\varphi, Q(\psi))\mu_\gamma +$$
$$\frac{\nu^2}{2}\left(-\langle\varphi, E\varphi\rangle + h(Riem(\varphi, d\psi)\varphi, d\psi)\right)\mu_\gamma + o(\nu^2)$$

where $h^{\alpha\beta}_{ab}(x) = \gamma^{\alpha\beta}(x) g_{ab}(\psi(x))$ while $E\varphi \doteq \mathrm{Tr}_h\left(\nabla^\psi \circ \nabla^\psi \varphi\right)$.

# Properties of the operator $E$

## Proposition

*The map $E : \Gamma(\psi^* TM) \to \Gamma(\psi^* TM)$ is a second order, elliptic partial differential operator whose principal symbol coincides with that of the operator $\widehat{E}$ such that*

$$\left(\widehat{E}\varphi\right)_a \doteq g_{ab}\left(\Delta_\gamma \varphi\right)^b ,$$

*where $\Delta_\gamma$ is the Laplace-Beltrami operator associated to $\gamma$.*

- $E$ admits a parametrix $P : \Gamma_c(\psi^* TM) \to \Gamma(\psi^* TM)$
- in any geodesically convex neighbourhood $\mathcal{O}$ it reads

$$P^{ab}(x,x') = H^{ab}(x,x') + W^{ab}(x,x'), \quad W \in \Gamma(S^{\boxtimes 2}\psi^* T\mathcal{O}),$$

$$H^{ab}(x,x') = g^{ab}(\psi(x),\psi(x'))v(x,x')\log\frac{\sigma(x,x')}{\lambda^2},\ v \in C^\infty(\mathcal{O}\times\mathcal{O})$$

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# The abstract formulation

Can we formulate NLSM in terms of algebraic QFT?

In view of our experience with theories on Lorentzian manifolds, we expect

- formulation of NLSM as a functor
  - from a category of background data encoding the geometry
  - to a category of *commutative* unital $*$-algebras – $P$ is symmetric
- observables with non-intersecting support must commute

We do not expect

- a time-slice axiom and an on-shell formulation
- existence of singularities except at coinciding points

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# Many thanks to ...

... all whose works helped creating the perfect tide

- Euclidean AQFT: Osterwalder, Schrader, Schlingemann, Wald....
  - K. Keller, J. Math. Phys. **50** (2009) 103503
- NLSM: Friedan, Gawedzki, Zahn....
  - D. Bahns, K. Rejzner & J. Zahn, CMP **327** (2014) 779
- Wick ordering: Brunetti, Fredenhagen, Hollands, Köhler, Wald....
  - I. Khavkine, A. Melati & V.Moretti, AHP **20** (2019) 929
- pAQFT: Brunetti, Duetsch, Fredenhagen, Rejzner....
  - R. Brunetti , M. Duetsch & K. Fredenhagen, Adv. Theor. Math. Phys. **13** (2009), 1541

# The Background data

We set

- $\mathfrak{BkgG}$ the category of background data:
  - *Objects* $\mathrm{Obj}(\mathfrak{BkgG})$ are pairs (N;b)
    - $N \equiv (\Sigma, M)$ connected, oriented manifolds with $\dim \Sigma = 2$ and $\dim M = D \geq 2$,
    - $b \equiv (\gamma, g, \psi)$ where $\gamma, g$ are Riemannian metrics on $\Sigma, M$ while $\psi \in C^\infty(\Sigma; M)$.
  - *Arrows* $\mathrm{Arr}(\mathfrak{BkgG})$ are pairs $(t, \tau)$ where
    - $(t, \tau)$ are orientation preserving isometric embeddings

$$\tau : \Sigma \to \Sigma', \quad \text{and} \quad t : M \to M'$$

    - subject to the constraint $\psi' \circ \tau = t \circ \psi$
- $\mathfrak{Alg}$ is the category of *(commutative)* $*$*-algebras*

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# Local Functionals

We call

$$\mathcal{P}[N;b] = \mathrm{span}_{\mathbb{C}}\left\{F : \Gamma(\psi^* TM) \to \mathbb{C}\right\}$$

generated by

$$F_{\omega_k}(\varphi) = \int_{\Sigma} \mu_\gamma \left\langle \varphi^k, \omega_k \right\rangle, \quad \omega_k \in \Gamma_c((\psi^* T^* M)^{\otimes k}),\ k \in \mathbb{N} \cup \{0\}$$

which are

- local $\forall \varphi \in \Gamma(\psi^* T^* M)$,

  $$\mathrm{supp}\left(F^{(n)}(\varphi)\right) \subset D_n \doteq \{(x,...,x) \in \Sigma^n\} \text{ and } WF(F^{(n)}(\varphi)) \perp T^* D_n,$$

- polynomial $F^{(n)}(\varphi) = 0$ for all $n > n_0$.

# The algebra

Fix any parametric of $E$, $P \in \mathrm{Par}[N; b]$. For any $F, G \in \mathcal{P}[N; b]$

$$(F \cdot_P G)(\varphi) = (\mathcal{M} \circ \exp[\Gamma_P](F \otimes G))(\varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle F^{(n)}(\varphi), P^{\otimes n} G^{(n)}(\varphi) \rangle.$$

- The product is well-defined and convergent (polynomial functionals)
- It is compatible with $F(\varphi)^* \doteq \overline{F(\varphi)}$

$(\mathcal{P}[N; b], \cdot_P, *)$ is a **unital *-algebra**

Problem: The choice of $P$ is **not** covariant

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# The bundle of algebras

Observe that, $\forall P, P' \in \mathrm{Par}[N; b]$ the map

$$\alpha_P^{P'} : \mathcal{P}[N; b] \to \mathcal{P}[N; b], \quad (\alpha_P^{P'} F)(\varphi) \doteq \sum_{n=0}^{\infty} \frac{1}{2^n n!} \langle (P - P')^n, F^{(2n)}(\varphi) \rangle$$

is a $*$-isomorphism.

- We call $\mathcal{E}[N; b]$ the bundle $\bigsqcup_{P \in \mathrm{Par}[M;b]} \mathcal{P}[N; b]$

$$\Gamma_{eq}(\mathcal{E}[N; b]) \doteq \{F \in \Gamma(\mathcal{E}[N; b]) \mid (\alpha_P^{P'} F)(P) = F(P') \; \forall P, P' \in \mathrm{Par}[N; b]\}$$

identifies a unital *-algebra $\mathcal{A}[N; b] = (\Gamma_{eq}(\mathcal{E}[N; b]), \cdot, *)$ where

$$(F \cdot G)(P) \doteq F(P) \cdot_P G(P), \quad F^*(P) \doteq F(P)^*$$

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# Locally covariant Euclidean field theory

In the spirit of R. Brunetti, K. Fredenhagen & R. Verch, CMP **237** (2003) 31

## Theorem

*The assignment* $\mathcal{A} : \mathfrak{Bkg}\mathfrak{G} \to \mathfrak{Alg}$ *such that*

1. $[N; b] \mapsto \mathcal{A}[N; b]$ *for all* $[N; b] \in Obj(\mathfrak{Bkg}\mathfrak{G})$
2. $\forall(\tau, t) \in Arr(\mathfrak{Bkg}\mathfrak{G})$, $\mathcal{A}[\tau, t] : \mathcal{A}[N; b] \to \mathcal{A}[N', b']$ *is such that*

$$(\mathcal{A}[\tau, t]F)(P, \varphi) \doteq F(\tau^* P, \tau^* \varphi).$$

*is a covariant functor.*

Observe that, in addition $\mathfrak{Bkg}\mathfrak{G}$ is **dimensionful**:

$$\forall \lambda > 0 \; \rho_\lambda : \mathfrak{Bkg}\mathfrak{G} \to \mathfrak{Bkg}\mathfrak{G} \quad \rho_\lambda[N; b] = [N; b_\lambda] \equiv (\Sigma, M; \psi, \lambda^{-2}\gamma, \lambda^{D-2} g)$$

is a *natural transformation*.

# Locally Covariant Observables

The *covariant functor of test-forms* $\Gamma^{\bullet,k} : \mathfrak{BkgG} \to \mathfrak{Alg}$, $k \in \mathbb{N}$ is

$$\Gamma_c^{\bullet,k}[N;b] = \bigoplus_{m=0}^{\infty} \Gamma_c(S^{\boxtimes m} S^{\otimes k} \psi^* T^* M), \quad \Gamma[\tau, t] = \tau^*.$$

## Definition

We call locally covariant observable (LCO) of degree $k$ a natural transformation $\mathcal{O}_k : \Gamma_c^{\bullet,k} \to \mathcal{A}$.

**Example:** $\Phi[N;b] : \Gamma^{\bullet,1}[N;b] \to \mathcal{A}[N;b]$

$$\Phi[N;b](\omega_1, P, \varphi) \doteq \int_\Sigma \mu_\gamma \langle \omega_1, \varphi \rangle \quad \omega_1 \in \Gamma_c(\psi^* T^* M)$$

is a locally covariant quantum field.

# The scaling behaviour

For every LCO $\mathcal{O}_k$ of degree $k$ and for all $\lambda > 0$

$$(S_\lambda \mathcal{O}_k)[N; b](\omega_m) \doteq \mathcal{O}_k[N; b](\lambda^{2m}\omega_m) \quad \omega_m \in \Gamma_c^{m,k}(\psi^* T^* M).$$

We say that $\mathcal{O}_k$ scales

- homogeneously with dimension $d \in \mathbb{R}$ if

$$(S_\lambda \mathcal{O}_k)[N; b](\omega_m) = \lambda^{dm}\mathcal{O}_k[N; b](\omega_m),$$

- almost homogeneously with dimension $d \in \mathbb{R}$ and order $\ell \in \mathbb{N}$ if

$$(S_\lambda \mathcal{O}_k)[N; b](\omega_m) = \lambda^{dm}\mathcal{O}_k[N; b](\omega_m) + \lambda^{dm}\sum_{j \leq \ell}(\log \lambda)^j \mathcal{O}_j[N; b](\omega_m).$$

Problem: We need Wick powers to read $\mathcal{L}_H$ as a LCO

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Problem: We need Wick powers to read $\mathcal{L}_H$ as a LCO

# Wick Powers

We call Wick powers a family of *natural transformations*

$$\Phi^\bullet = \{\Phi^k\}_{k\in\mathbb{N}}, \quad \Phi^k : \Gamma_c^{\bullet,k} \to \mathcal{A}$$

1. $\forall k \in \mathbb{N}$, $\Phi^k$ is a LCO which scales almost homog. with dimension $d = 0$,
2. $\Phi^0 = id_{\mathcal{A}}$ and $\Phi^1 = \Phi$,
3. for all possible choices of the arguments

$$\langle \Phi^k[N;b](\omega_1, P, \varphi_1)^{(1)}, \varphi_2 \rangle = k\Phi^{k-1}[N;b](\varphi_2 \lrcorner \omega_1, P, \varphi_1),$$

4. a technical condition on the WF of each $\Phi^k[N;b]$

Goal: prove existence of Wick powers and classify the freedom in their construction

# Wick Powers – Ambiguities

Peetre-Slóvak theorem and the work of Khavkine, Melati & Moretti imply

## Theorem

*Let $\Phi^\bullet$ and $\widehat{\Phi}^\bullet$ two families of Wick powers. There exists*

$$\{C_\ell\}_{\ell=1}^{k-2} : \Gamma_c^{\bullet,\ell} \to \mathcal{A} \quad C_\ell[N;b](\omega_1) = \int_\Sigma d\mu_\gamma \langle D_\ell(\gamma, \psi^* g), \omega_1 \rangle id|_{\mathcal{A}[N;b]}$$

*which are LCOs scaling almost homog. with degree $d = 0$ and*

$$D_\ell : \Gamma(S^{\otimes 2} T^*\Sigma \otimes S^{\otimes 2} \psi^* T^* M) \to \Gamma(S^{\otimes \ell} \psi^* T^* M)$$

*which are differential operators of globally bounded order, tensorially and polynomially constructed from their arguments. Finally*

$$\widehat{\Phi}^k[N;b](\omega_1) = \Phi^k[N;b](\omega_1) + \sum_{\ell=0}^{k-2} \Phi^\ell[N;b](D_\ell[N;b] \lrcorner \omega_1).$$

# Expanding the Lagrangian

We split the Lagrangian in a free and in an interacting part

$$\mathcal{L}_{\text{free}}(\psi,\gamma,g;\varphi) = -\frac{\nu^2}{2}\langle\varphi, E\varphi\rangle\mu_\gamma$$
$$\mathcal{L}_{\text{int}}(\psi,\gamma,g;\varphi) = \mathcal{L}_H[\psi,\gamma,g] + \left(\nu g\left(\varphi, Q\psi\right) + \frac{\nu^2}{2} h\left(\text{Riem}(\varphi, d\psi)\varphi, d\psi\right)\right)\mu_\gamma$$

**Proposition**

*The interacting Lagrangian $\mathcal{L}_{int}$ identifies a Locally Covariant Observable $\mathcal{L}_{int}[\Phi^\bullet]$*

# Renormalization Group Flow - I

Consider a second family of Wick powers $\widehat{\Phi}^\bullet = S_\lambda \Phi^\bullet$, $\lambda > 0$

Since

1. $\mathcal{L}_{\mathrm{int}}$ is a Locally Covariant Observable,
2. we classified the difference between two families of Wick powers,

$$\mathcal{L}_{\mathrm{int}}[S_\lambda \Phi^\bullet] = \mathcal{L}_{\mathrm{int}}[\Phi^\bullet] + R_\lambda[\Phi^\bullet].$$

Mantra of Renormalization: Reabsorb $R_\lambda[\Phi^\bullet]$ in the coupling constants:

$$\mathcal{L}_{\mathrm{int}}[S_\lambda \Phi^\bullet] = \mathcal{L}_{\mathrm{int},\lambda}[\Phi^\bullet].$$

# Renormalization Group Flow - I

Consider a second family of Wick powers $\widehat{\Phi}^\bullet = S_\lambda \Phi^\bullet$, $\lambda > 0$

Since

1. $\mathcal{L}_{\mathrm{int}}$ is a Locally Covariant Observable,
2. we classified the difference between two families of Wick powers,

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Mantra of Renormalization: Reabsorb $R_\lambda[\Phi^\bullet]$ in the coupling constants:

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# Renormalization Group Flow - II

## Theorem (Ricci Flow)

*With the previous notations*

$$\mathcal{L}_\lambda[\Phi^\bullet][N;b] = \mathcal{L}_{free}[N;b] + \mathcal{L}_{int,\lambda}[N;b]$$

*where*

$$\mathcal{L}_\lambda[\Phi^\bullet][N;b] = Tr_\gamma(\psi^* g_{\log\lambda})\mu_\gamma +$$
$$\left(\nu g(\varphi, Q(\psi)) + \frac{\nu^2}{2}\left(-\langle\varphi, E\varphi\rangle + h(Riem(\varphi, d\psi)\varphi, d\psi)\right)\right)\mu_\gamma.$$

*Here* $g_{\log\lambda} = g + \nu^2 \log(\lambda) Ric[g]$. *Setting* $\lambda = e^{-2\beta}$ *and* $g(\beta) \doteq g_{-2\beta}$

$$\frac{dg(\beta)}{d\beta} = -2\nu^2 Ric[g] = -2\nu^2 Ric[g(\beta)] + o(\nu^2).$$

# And now?

Outlook

- We have developed a general framework to discuss Euclidean AQFT
- We have constructed the algebra of Wick powers for nonlinear Sigma models
- We have given a proof of the emergence of Ricci flow

To do

- Compute higher order flows (RG-2 flow)
- Develop pAQFT to its fullest extent for Euclidean theories
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