Ricci Flow from Euclidean AQFT

Claudio Dappiaggi
DPG Tagung
München – 18/03/2019
Dipartimento di Fisica – University of Pavia
Outline of the Talk

1. Motivations and the Ricci Flow
2. Nonlinear Sigma models
3. Euclidean AQFT
4. Non Linear Sigma Models as EAQFT
5. Ricci flow and Renormalization

Based on

The AQFT viewpoint

Algebraic Quantum Field Theory is a two-step approach:

- assign a unital $\ast$-algebra $\mathcal{A}$ encoding CCRs, causality, time-slice axiom, etc.
- assign a state $\omega : \mathcal{A} \rightarrow \mathbb{C}$

\[ \omega(\mathbb{I}) = 1, \quad \omega(a^*a) \geq 0, \quad \forall a \in \mathcal{A} \]

It is a very effective and rigorous approach to analyze

- QFT on Lorentzian curved backgrounds,
- conformal field theories, especially in 2D,
- gauge theories (BRST, BV, ....),
- interacting models form a perturbative viewpoint (pAQFT),
- regularization and renormalization....

Can we enlarge the realm of applications of AQFT? Is it useful?
Motivations

The AQFT viewpoint

Algebraic Quantum Field Theory is a two-step approach:

- assign a unital \(*\)-algebra \( \mathcal{A} \) encoding CCRs, causality, time-slice axiom,....
- assign a state \( \omega : \mathcal{A} \rightarrow \mathbb{C} \)

\[ \omega(\mathbb{I}) = 1, \quad \omega(a^*a) \geq 0, \quad \forall a \in \mathcal{A} \]

It is a very effective and rigorous approach to analyze

- QFT on Lorentzian curved backgrounds,
- conformal field theories, especially in 2D,
- gauge theories (BRST, BV, ....),
- interacting models form a perturbative viewpoint (pAQFT),
- regularization and renormalization....

Can we enlarge the realm of applications of AQFT? Is it useful?
Motivations

The AQFT viewpoint

Algebraic Quantum Field Theory is a two-step approach:

- assign a **unital** *-algebra $\mathcal{A}$ encoding CCRs, causality, time-slice axiom,....
- assign a state $\omega : \mathcal{A} \to \mathbb{C}$

$$\omega(\mathbb{1}) = 1, \quad \omega(a^*a) \geq 0, \quad \forall a \in \mathcal{A}$$

It is a very effective and rigorous approach to analyze

- QFT on *Lorentzian* curved backgrounds,
- conformal field theories, especially in 2D,
- gauge theories (BRST, BV, ....),
- interacting models form a perturbative viewpoint (pAQFT),
- regularization and renormalization....

Can we enlarge the realm of applications of AQFT? Is it useful?
Motivations

The AQFT viewpoint

Algebraic Quantum Field Theory is a two-step approach:

- assign a unital \(*\)-algebra \(\mathcal{A}\) encoding CCRs, causality, time-slice axiom,....
- assign a state \(\omega : \mathcal{A} \rightarrow \mathbb{C}\)

\[
\omega(I) = 1, \quad \omega(a^*a) \geq 0, \quad \forall a \in \mathcal{A}
\]

It is a very effective and rigorous approach to analyze

- QFT on Lorentzian curved backgrounds,
- conformal field theories, especially in 2D,
- gauge theories (BRST, BV, ....),
- interacting models form a perturbative viewpoint (pAQFT),
- regularization and renormalization....

Can we enlarge the realm of applications of AQFT? Is it useful?
Motivations

The AQFT viewpoint

Algebraic Quantum Field Theory is a two-step approach:

- assign a unital *-algebra $\mathcal{A}$ encoding CCRs, causality, time-slice axiom,....
- assign a state $\omega : \mathcal{A} \rightarrow \mathbb{C}$

$$\omega(\mathbb{I}) = 1, \quad \omega(a^* a) \geq 0, \quad \forall a \in \mathcal{A}$$

It is a very effective and rigorous approach to analyze

- QFT on Lorentzian curved backgrounds,
- conformal field theories, especially in 2D,
- gauge theories (BRST, BV, ....),
- interacting models form a perturbative viewpoint (pAQFT),
- regularization and renormalization....

Can we enlarge the realm of applications of AQFT? Is it useful?
Motivations

The AQFT viewpoint

Algebraic Quantum Field Theory is a two-step approach:

- assign a **unital** \(*\)-algebra \(\mathcal{A}\) encoding CCRs, causality, time-slice axiom, ....
- assign a **state** \(\omega : \mathcal{A} \rightarrow \mathbb{C}\)

\[
\omega(\mathbb{I}) = 1, \quad \omega(a^*a) \geq 0, \quad \forall a \in \mathcal{A}
\]

It is a very effective and rigorous approach to analyze

- QFT on *Lorentzian* curved backgrounds,
- conformal field theories, especially in 2D,
- gauge theories (BRST, BV, ....),
- interacting models form a perturbative viewpoint (pAQFT),
- regularization and renormalization....

Can we enlarge the realm of applications of AQFT? Is it useful?
Motivations

The AQFT viewpoint

Algebraic Quantum Field Theory is a two-step approach:

- assign a unital *-algebra $\mathcal{A}$ encoding CCRs, causality, time-slice axiom,....
- assign a state $\omega : \mathcal{A} \to \mathbb{C}$

$$\omega(\mathbb{I}) = 1, \quad \omega(a^*a) \geq 0, \quad \forall a \in \mathcal{A}$$

It is a very effective and rigorous approach to analyze

- QFT on Lorentzian curved backgrounds,
- conformal field theories, especially in 2D,
- gauge theories (BRST, BV, ....),
- interacting models form a perturbative viewpoint (pAQFT),
- regularization and renormalization....

Can we enlarge the realm of applications of AQFT? Is it useful?
Motivations

Ricci Flow

Let \((M, g_0)\) be a Riemannian manifold and \(I \subseteq \mathbb{R} \ni \beta \mapsto g(\beta) \in \text{Riem}(M)\)

\[
\frac{\partial g(\beta)}{\partial \beta} = -2Ric[g(\beta)], \quad g(0) = g_0
\]

This is the Ricci Flow

It has great relevance:

1. In geometric analysis as a key ingredient in the proof of Thurston geometrization conjecture (Hamilton – 1982 & Perelman – 2003)
2. In theoretical physics as the renormalization group flow of nonlinear Sigma models (D. Friedan – Ann. of Phys. 163 (1985) 1057)
3. In averaging cosmology – see M. Carfora’s talk SYKM 1.2 – Tue. 17:10

Problem: many derivations but all with some limitations...
Motivations

Ricci Flow

Let \((M, g_0)\) be a \textbf{Riemannian manifold} and \(I \subseteq \mathbb{R} \ni \beta \mapsto g(\beta) \in \text{Riem}(M)\)

\[
\frac{\partial g(\beta)}{\partial \beta} = -2\text{Ric}[g(\beta)], \quad g(0) = g_0
\]

This is the \textbf{Ricci Flow}

It has great relevance:

1. In geometric analysis as a key ingredient in the proof of Thurston geometrization conjecture (Hamilton – 1982 & Perelman – 2003)
2. In theoretical physics as the \textit{renormalization group flow} of nonlinear Sigma models (D. Friedan – Ann. of Phys. \textbf{163} (1985) 1057)
3. In averaging cosmology – see M. Carfora’s talk SYKM 1.2 – Tue. 17:10

Problem: many derivations but all with some limitations...
Motivations

Ricci Flow

Let \((M, g_0)\) be a Riemannian manifold and \(I \subseteq \mathbb{R} \ni \beta \mapsto g(\beta) \in \text{Riem}(M)\)

\[
\frac{\partial g(\beta)}{\partial \beta} = -2 \text{Ric}[g(\beta)], \quad g(0) = g_0
\]

This is the Ricci Flow

It has great relevance:

1. In geometric analysis as a key ingredient in the proof of Thurston geometrization conjecture (Hamilton – 1982 & Perelman – 2003)

2. In theoretical physics as the renormalization group flow of nonlinear Sigma models (D. Friedan – Ann. of Phys. 163 (1985) 1057)

3. In averaging cosmology – see M. Carfora’s talk SYKM 1.2 – Tue. 17:10

Problem: many derivations but all with some limitations...
Let \((M, g_0)\) be a Riemannian manifold and \(I \subseteq \mathbb{R} \ni \beta \mapsto g(\beta) \in \text{Riem}(M)\)

\[
\frac{\partial g(\beta)}{\partial \beta} = -2Ric[g(\beta)], \quad g(0) = g_0
\]

This is the Ricci Flow

It has great relevance:

1. In geometric analysis as a key ingredient in the proof of Thurston geometrization conjecture (Hamilton – 1982 & Perelman – 2003)
2. In theoretical physics as the renormalization group flow of nonlinear Sigma models (D. Friedan – Ann. of Phys. 163 (1985) 1057)
3. In averaging cosmology – see M. Carfora’s talk SYKM 1.2 – Tue. 17:10

Problem: many derivations but all with some limitations...
Nonlinear Sigma Models

Main Ingredients:

- \((\Sigma, \gamma)\) a (connected, oriented) Riemannian manifold \(\text{dim} \Sigma = 2\),
- \((M, g)\) a Riemannian manifold with \(\text{dim} M = D \geq 2\),
- \(\psi \in C^\infty(\Sigma; M)\)

Dynamics is encoded in the Lagrangian density

\[
\mathcal{L}_H[\gamma, g, \psi] = \text{Tr}_\gamma(\psi^* g)_{\mu \gamma} \equiv g_{ab}(\psi(x))^{\gamma^a \beta} d\psi^a \alpha d\psi^b_{\beta}
\]

Next step: Expand to 2nd order!
Nonlinear Sigma Models

Main Ingredients:

- \((\Sigma, \gamma)\) a (connected, oriented) Riemannian manifold \(\dim \Sigma = 2\),
- \((M, g)\) a Riemannian manifold with \(\dim M = D \geq 2\),
- \(\psi \in C^\infty(\Sigma; M)\)

Dynamics is encoded in the Lagrangian density

\[
L_H[\gamma, g, \psi] = \text{Tr}_\gamma(\psi^* g) \mu_\gamma \equiv g_{ab}(\psi(x)) \gamma^{\alpha\beta} d\psi^a_\alpha d\psi^b_\beta
\]

Next step: Expand to 2nd order!
Nonlinear Sigma models

NLSM – expansion to 2nd order

Kinematical configurations: $\Gamma(\psi^* TM) \ni \varphi$

- Via the exponential map $\exp_{\psi(x)} : T_{\psi(x)} M \to M$

\[ \psi_\nu(x) \doteq \exp_{\psi(x)}(\nu \varphi(x)). \]

The Lagrangian is expanded as:

\[ \mathcal{L}_H[\gamma, g, \psi_\nu] = \mathcal{L}_H[\gamma, g, \psi] + \nu g(\varphi, Q(\psi)) \mu_\gamma + \frac{\nu^2}{2} (\langle \varphi, E\varphi \rangle + h(Riem(\varphi, d\psi) \varphi, d\psi)) \mu_\gamma + o(\nu^2) \]

where $h_{ab}^{\alpha \beta}(x) = \gamma^{\alpha \beta}(x) g_{ab}(\psi(x))$ while $E\varphi \doteq \text{Tr}_h (\nabla^\psi \circ \nabla^\psi \varphi)$. 
Nonlinear Sigma models

Properties of the operator $E$

**Proposition**

The map $E : \Gamma(\psi^* TM) \to \Gamma(\psi^* TM)$ is a second order, elliptic partial differential operator whose principal symbol coincides with that of the operator $\hat{E}$ such that

$$\left( \hat{E} \varphi \right)_a = g_{ab} (\Delta_\gamma \varphi)^b,$$

where $\Delta_\gamma$ is the Laplace-Beltrami operator associated to $\gamma$.

- $E$ admits a parametrix $P : \Gamma_c(\psi^* TM) \to \Gamma(\psi^* TM)$
- in any geodesically convex neighbourhood $\mathcal{O}$ it reads

  $$P^{ab}(x, x') = H^{ab}(x, x') + W^{ab}(x, x'), \quad W \in \Gamma(S^{\otimes 2} \psi^* T\mathcal{O}),$$

  $$H^{ab}(x, x') = g^{ab}(\psi(x), \psi(x')) v(x, x') \log \frac{\sigma(x, x')}{\lambda^2}, \quad v \in C_\infty(\mathcal{O} \times \mathcal{O})$$
Proposition

The map \( E : \Gamma(\psi^* TM) \to \Gamma(\psi^* TM) \) is a second order, elliptic partial differential operator whose principal symbol coincides with that of the operator \( \hat{E} \) such that

\[
\left( \hat{E} \varphi \right)_a \equiv g_{ab} (\Delta_\gamma \varphi)^b ,
\]

where \( \Delta_\gamma \) is the Laplace-Beltrami operator associated to \( \gamma \).

- \( E \) admits a parametrix \( P : \Gamma_c(\psi^* TM) \to \Gamma(\psi^* TM) \)
- in any geodesically convex neighbourhood \( \mathcal{O} \) it reads

\[
P^{ab}(x, x') = H^{ab}(x, x') + W^{ab}(x, x'), \quad W \in \Gamma(S^{\boxtimes 2} \psi^* T\mathcal{O}),
\]

\[
H^{ab}(x, x') = g^{ab}(\psi(x), \psi(x')) v(x, x') \log \frac{\sigma(x, x')}{\chi^2}, \quad v \in C^\infty(\mathcal{O} \times \mathcal{O})
\]
Can we formulate NLSM in terms of algebraic QFT?

In view of our experience with theories on Lorentzian manifolds, we expect

- formulation of NLSM as a functor
  - from a category of background data encoding the geometry
  - to a category of commutative unital $\ast$-algebras – $P$ is symmetric
  - observables with non-intersecting support must commute

We do not expect

- a time-slice axiom and an on-shell formulation
- existence of singularities except at coinciding points
Can we formulate NLSM in terms of algebraic QFT?

In view of our experience with theories on Lorentzian manifolds, we expect:

- formulation of NLSM as a functor
  - from a category of background data encoding the geometry
  - to a category of commutative unital $\ast$-algebras – $P$ is symmetric
- observables with non-intersecting support must commute

We do not expect:

- a time-slice axiom and an on-shell formulation
- existence of singularities except at coinciding points
Many thanks to ...

... all whose works helped creating the perfect tide

- Euclidean AQFT: Osterwalder, Schrader, Schlingemann, Wald....

- NLSM: Friedan, Gawedzki, Zahn....
  - D. Bahns, K. Rejzner & J. Zahn, CMP 327 (2014) 779

- Wick ordering: Brunetti, Fredenhagen, Hollands, K"ohler, Wald....
  - I. Khavkine, A. Melati & V. Moretti, AHP 20 (2019) 929

- pAQFT: Brunetti, Duetsch, Fredenhagen, Rejzner....
We set

- $\mathcal{BG}$ the category of **background data**:

  - **Objects** $\text{Obj}(\mathcal{BG})$ are pairs $(N; b)$
    - $N \equiv (\Sigma, M)$ connected, oriented manifolds with $\dim \Sigma = 2$ and $\dim M = D \geq 2$,
    - $b \equiv (\gamma, g, \psi)$ where $\gamma, g$ are Riemannian metrics on $\Sigma, M$ while $\psi \in C^\infty(\Sigma; M)$.

  - **Arrows** $\text{Arr}(\mathcal{BG})$ are pairs $(t, \tau)$ where
    - $(t, \tau)$ are orientation preserving isometric embeddings
      \[ \tau : \Sigma \to \Sigma', \quad \text{and} \quad t : M \to M' \]
    - subject to the constraint $\psi' \circ \tau = t \circ \psi$

- $\mathcal{Alg}$ is the category of (commutative) $*$-algebras
We set

- $\mathfrak{BkgG}$ the category of background data:
  - **Objects** $\text{Obj}(\mathfrak{BkgG})$ are pairs $(N;b)$
    - $N \equiv (\Sigma, M)$ connected, oriented manifolds with $\dim \Sigma = 2$ and $\dim M = D \geq 2$,
    - $b \equiv (\gamma, g, \psi)$ where $\gamma, g$ are Riemannian metrics on $\Sigma, M$ while $\psi \in C^\infty(\Sigma; M)$.
  - **Arrows** $\text{Arr}(\mathfrak{BkgG})$ are pairs $(t, \tau)$ where
    - $(t, \tau)$ are orientation preserving isometric embeddings
      \[ \tau : \Sigma \to \Sigma', \quad \text{and} \quad t : M \to M' \]
    - subject to the constraint $\psi' \circ \tau = t \circ \psi$

- $\text{Alg}$ is the category of (commutative) $\ast$-algebras
We call
\[ \mathcal{P}[N;b] = \text{span}_\mathbb{C} \{ F : \Gamma(\psi^* TM) \to \mathbb{C} \} \]
generated by
\[ F_{\omega_k}(\varphi) = \int_\Sigma \mu_\gamma \langle \varphi^k, \omega_k \rangle, \quad \omega_k \in \Gamma_c((\psi^* T^* M)^\otimes k), \quad k \in \mathbb{N} \cup \{0\} \]
which are
- local \( \forall \varphi \in \Gamma(\psi^* T^* M) \),
- polynomial \( F^{(n)}(\varphi) = 0 \) for all \( n > n_0 \).
Nonlinear Sigma models as Euclidean AQFT

The algebra

Fix any parametric of \( E, P \in \text{Par}[N; b] \). For any \( F, G \in \mathcal{P}[N; b] \)

\[
(F \mathbin{\cdot}_P G)(\varphi) = (\mathcal{M} \circ \exp[\Gamma_P](F \otimes G))(\varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle F^{(n)}(\varphi), P^\otimes n G^{(n)}(\varphi) \rangle.
\]

- The product is well-defined and convergent (polynomial functionals)
- It is compatible with \( F(\varphi)^* \doteq \overline{F(\varphi)} \)

\((\mathcal{P}[N; b], \mathbin{\cdot}_P, \ast)\) is a \textbf{unital} \(*\)-\textbf{algebra}

Problem: The choice of \( P \) is not covariant
Fix any parametric of $E, P \in \text{Par}[N; b]$. For any $F, G \in \mathcal{P}[N; b]$

$$(F \cdot_P G)(\varphi) = (\mathcal{M} \circ \exp[\Gamma_P](F \otimes G))(\varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle F^{(n)}(\varphi), P \otimes^n G^{(n)}(\varphi) \rangle.$$ 

- The product is well-defined and convergent (polynomial functionals)
- It is compatible with $F(\varphi)^* \equiv \overline{F(\varphi)}$

$(\mathcal{P}[N; b], \cdot_P, \ast)$ is a unital $\ast$-algebra

Problem: The choice of $P$ is not covariant
Observe that, $\forall P, P' \in \text{Par}[N; b]$ the map

$$\alpha^{P'}_P : \mathcal{P}[N; b] \to \mathcal{P}[N; b], \quad (\alpha^{P'}_P F)(\varphi) \doteq \sum_{n=0}^{\infty} \frac{1}{2^n n!} \langle (P - P')^n, F^{(2n)}(\varphi) \rangle$$

is a $\ast$-isomorphism.

- We call $\mathcal{E}[N; b]$ the bundle $\bigsqcup_{P \in \text{Par}[M; b]} \mathcal{P}[N; b]$

$$\Gamma_{\text{eq}}(\mathcal{E}[N; b]) \doteq \{ F \in \Gamma(\mathcal{E}[N; b]) \mid (\alpha^{P'}_P F)(P) = F(P') \forall P, P' \in \text{Par}[N; b] \}$$

identifies a unital $\ast$-algebra $\mathcal{A}[N; b] = (\Gamma_{\text{eq}}(\mathcal{E}[N; b]), \cdot, \ast)$ where

$$(F \cdot G)(P) \doteq F(P) \cdot_P G(P), \quad F^\ast(P) \doteq F(P)^\ast$$
Nonlinear Sigma models as Euclidean AQFT

The bundle of algebras

Observe that, \( \forall P, P' \in \text{Par}[N; b] \) the map

\[
\alpha_P^{P'} : \mathcal{P}[N; b] \to \mathcal{P}[N; b], \quad (\alpha_P^{P'} F)(\varphi) \doteq \sum_{n=0}^{\infty} \frac{1}{2^n n!} \langle (P - P')^n, F^{(2n)}(\varphi) \rangle
\]

is a *-isomorphism.

- We call \( \mathcal{E}[N; b] \) the bundle

\[
\bigsqcup_{P \in \text{Par}[M; b]} \mathcal{P}[N; b]
\]

\[
\Gamma_{eq}(\mathcal{E}[N; b]) \doteq \{ F \in \Gamma(\mathcal{E}[N; b]) \mid (\alpha_P^{P'} F)(P) = F(P') \ \forall P, P' \in \text{Par}[N; b] \}
\]

identifies a **unital** *-algebra \( \mathcal{A}[N; b] = (\Gamma_{eq}(\mathcal{E}[N; b]), \cdot, \ast) \) where

\[
(F \cdot G)(P) \doteq F(P) \cdot_P G(P), \quad F^*(P) \doteq F(P)^*
\]
Nonlinear Sigma models as Euclidean AQFT

Locally covariant Euclidean field theory


Theorem

The assignment $\mathcal{A} : \mathfrak{BkgG} \to \mathfrak{Alg}$ such that

1. $[N; b] \mapsto \mathcal{A}[N; b]$ for all $[N; b] \in \text{Obj}(\mathfrak{BkgG})$

2. $\forall (\tau, t) \in \text{Arr}(\mathfrak{BkgG})$, $\mathcal{A}[\tau, t] : \mathcal{A}[N; b] \to \mathcal{A}[N', b']$ is such that

$$ (\mathcal{A}[\tau, t] F)(P, \varphi) \doteq F(\tau^* P, \tau^* \varphi). $$

is a covariant functor.

Observe that, in addition $\mathfrak{BkgG}$ is dimensionful:

$$ \forall \lambda > 0 \; \rho_{\lambda} : \mathfrak{BkgG} \to \mathfrak{BkgG} \; \rho_{\lambda}[N; b] = [N; b_{\lambda}] \equiv (\Sigma, M; \psi, \lambda^{-2} \gamma, \lambda^{D-2} g) $$

is a natural transformation.
The covariant functor of test-forms $\Gamma^{\bullet,k} : \mathcal{BkgG} \to \mathcal{Alg}, \ k \in \mathbb{N}$ is

$$\Gamma^{\bullet,k}[N; b] = \bigoplus_{m=0}^{\infty} \Gamma_c(S \boxtimes^m S \otimes^k \psi^* T^* M), \quad \Gamma[\tau, t] = \tau^*.$$  

**Definition**

We call locally covariant observable (LCO) of degree $k$ a natural transformation $O_k : \Gamma^{\bullet,k} \to \mathcal{A}$.

**Example:** $\Phi[N; b] : \Gamma^{\bullet,1}[N; b] \to \mathcal{A}[N; b]$

$$\Phi[N; b](\omega_1, P, \varphi) \doteq \int_{\Sigma} \mu \langle \omega_1, \varphi \rangle \quad \omega_1 \in \Gamma_c(\psi^* T^* M)$$

is a locally covariant quantum field.
The scaling behaviour

For every LCO $\mathcal{O}_k$ of degree $k$ and for all $\lambda > 0$

$$(S_\lambda \mathcal{O}_k)[N; b](\omega_m) \doteq \mathcal{O}_k[N; b](\lambda^{2m} \omega_m) \quad \omega_m \in \Gamma^m_{c,k}(\psi^* T^* M).$$

We say that $\mathcal{O}_k$ scales

- **homogeneously** with dimension $d \in \mathbb{R}$ if

  $$(S_\lambda \mathcal{O}_k)[N; b](\omega_m) = \lambda^{dm} \mathcal{O}_k[N; b](\omega_m),$$

- **almost homogeneously** with dimension $d \in \mathbb{R}$ and order $\ell \in \mathbb{N}$ if

  $$(S_\lambda \mathcal{O}_k)[N; b](\omega_m) = \lambda^{dm} \mathcal{O}_k[N; b](\omega_m) + \lambda^{dm} \sum_{j \leq \ell} (\log \lambda)^j \mathcal{O}_j[N; b](\omega_m).$$

Problem: We need Wick powers to read $\mathcal{L}_H$ as a LCO
The scaling behaviour

For every LCO $\mathcal{O}_k$ of degree $k$ and for all $\lambda > 0$

$$(S_\lambda \mathcal{O}_k)[N; b](\omega_m) \doteq \mathcal{O}_k[N; b](\lambda^{2m}\omega_m) \quad \omega_m \in \Gamma_{c}^{m,k}(\psi^* T^* M).$$

We say that $\mathcal{O}_k$ scales

- **homogeneously** with dimension $d \in \mathbb{R}$ if

  $$(S_\lambda \mathcal{O}_k)[N; b](\omega_m) = \lambda^{dm} \mathcal{O}_k[N; b](\omega_m),$$

- **almost homogeneously** with dimension $d \in \mathbb{R}$ and order $\ell \in \mathbb{N}$ if

  $$(S_\lambda \mathcal{O}_k)[N; b](\omega_m) = \lambda^{dm} \mathcal{O}_k[N; b](\omega_m) + \lambda^{dm} \sum_{j \leq \ell} (\log \lambda)^j \mathcal{O}_j[N; b](\omega_m).$$

**Problem:** We need Wick powers to read $\mathcal{L}_H$ as a LCO.
The scaling behaviour

For every LCO $\mathcal{O}_k$ of degree $k$ and for all $\lambda > 0$

$$\left( S_\lambda \mathcal{O}_k \right)[N; b](\omega_m) = \mathcal{O}_k[N; b](\lambda^{2m}\omega_m) \quad \omega_m \in \Gamma_{c,m,k}^m(\psi^* T^* M).$$

We say that $\mathcal{O}_k$ scales

- **homogeneously** with dimension $d \in \mathbb{R}$ if

  $$\left( S_\lambda \mathcal{O}_k \right)[N; b](\omega_m) = \lambda^{dm} \mathcal{O}_k[N; b](\omega_m),$$

- **almost homogeneously** with dimension $d \in \mathbb{R}$ and order $\ell \in \mathbb{N}$ if

  $$\left( S_\lambda \mathcal{O}_k \right)[N; b](\omega_m) = \lambda^{dm} \mathcal{O}_k[N; b](\omega_m) + \lambda^{dm} \sum_{j \leq \ell} (\log \lambda)^j \mathcal{O}_j[N; b](\omega_m).$$

**Problem:** We need Wick powers to read $\mathcal{L}_H$ as a LCO
We call Wick powers a family of *natural transformations* 

\[ \Phi^\bullet = \{\Phi^k\}_{k \in \mathbb{N}}, \quad \Phi^k : \Gamma_{c,k}^\bullet \to \mathcal{A} \]

1. \( \forall k \in \mathbb{N}, \Phi^k \) is a LCO which scales almost homog. with dimension \( d = 0 \),
2. \( \Phi^0 = id_A \) and \( \Phi^1 = \Phi \),
3. for all possible choices of the arguments

\[ \langle \Phi^k[N; b](\omega_1, P, \varphi_1^{(1)}, \varphi_2) = k\Phi^{k-1}[N; b](\varphi_2 \downarrow \omega_1, P, \varphi_1), \]

4. a technical condition on the WF of each \( \Phi^k[N; b] \)

**Goal:** prove existence of Wick powers and classify the freedom in their construction
Peetre-Slovák theorem and the work of Khavkine, Melati & Moretti imply

**Theorem**

Let $\Phi^\bullet$ and $\widehat{\Phi}^\bullet$ two families of Wick powers. There exists

$$\{ C_\ell \}_{\ell = 1}^{k-2} : \Gamma^\bullet_{c, \ell} \to \mathcal{A} \quad C_\ell[N; b](\omega_1) = \int_{\Sigma} d\mu_{\gamma} \langle D_\ell(\gamma, \psi^* g), \omega_1 \rangle id|_{\mathcal{A}[N; b]}$$

which are LCOs scaling almost homog. with degree $d = 0$ and

$$D_\ell : \Gamma(S^{\otimes 2} T^* \Sigma \otimes S^{\otimes 2} \psi^* T^* M) \to \Gamma(S^{\otimes \ell} \psi^* T^* M)$$

which are differential operators of globally bounded order, tensorially and polynomially constructed from their arguments. Finally

$$\widehat{\Phi}^k[N; b](\omega_1) = \Phi^k[N; b](\omega_1) + \sum_{\ell=0}^{k-2} \Phi^\ell[N; b](D_\ell[N; b] \omega_1).$$
We split the Lagrangian in a free and in an interacting part

\[ L_{\text{free}}(\psi, \gamma, g; \varphi) = -\frac{\nu^2}{2} \langle \varphi, E\varphi \rangle \mu_\gamma \]
\[ L_{\text{int}}(\psi, \gamma, g; \varphi) = L_H[\psi, \gamma, g] + \left( \nu g(\varphi, Q\psi) + \frac{\nu^2}{2} h(\text{Riem}(\varphi, d\psi)\varphi, d\psi) \right) \mu_\gamma \]

**Proposition**

The interacting Lagrangian \( L_{\text{int}} \) identifies a *Locally Covariant Observable* \( L_{\text{int}}[\Phi^*] \)
Consider a second family of Wick powers $\hat{\Phi} = S\lambda \Phi^\bullet$, $\lambda > 0$.

Since

1. $\mathcal{L}_{\text{int}}$ is a Locally Covariant Observable,
2. we classified the difference between two families of Wick powers,

$$\mathcal{L}_{\text{int}}[S\lambda \Phi^\bullet] = \mathcal{L}_{\text{int}}[\Phi^\bullet] + R_\lambda[\Phi^\bullet].$$

Mantra of Renormalization: Reabsorb $R_\lambda[\Phi^\bullet]$ in the coupling constants:

$$\mathcal{L}_{\text{int}}[S\lambda \Phi^\bullet] = \mathcal{L}_{\text{int},\lambda}[\Phi^\bullet].$$
Consider a second family of Wick powers $\hat{\Phi} = S_\lambda \Phi$, $\lambda > 0$.

Since

1. $\mathcal{L}_{\text{int}}$ is a Locally Covariant Observable,
2. we classified the difference between two families of Wick powers,

$$\mathcal{L}_{\text{int}}[S_\lambda \Phi] = \mathcal{L}_{\text{int}}[\Phi] + R_\lambda[\Phi].$$

**Mantra of Renormalization**: Reabsorb $R_\lambda[\Phi]$ in the coupling constants:

$$\mathcal{L}_{\text{int}}[S_\lambda \Phi] = \mathcal{L}_{\text{int},\lambda}[\Phi].$$
**Theorem (Ricci Flow)**

*With the previous notations*

\[
\mathcal{L}_\lambda[\Phi^\bullet][N; b] = \mathcal{L}_{\text{free}}[N; b] + \mathcal{L}_{\text{int,}\lambda}[N; b]
\]

*where*

\[
\mathcal{L}_\lambda[\Phi^\bullet][N; b] = Tr_\gamma(\psi^* g_{\log \lambda}) \mu_\gamma +
\left(\nu g(\varphi, Q(\psi)) + \frac{\nu^2}{2} (-\langle \varphi, E \varphi \rangle + h(Riem(\varphi, d\psi)\varphi, d\psi))\right) \mu_\gamma.
\]

*Here* \(g_{\log \lambda} = g + \nu^2 \log(\lambda) Ric[g].\) *Setting* \(\lambda = e^{-2\beta}\) *and* \(g(\beta) \triangleq g_{-2\beta}\)

\[
\frac{dg(\beta)}{d\beta} = -2\nu^2 Ric[g] = -2\nu^2 Ric[g(\beta)] + o(\nu^2).
\]
Conclusions

And now?

Outlook

- We have developed a general framework to discuss Euclidean AQFT
- We have constructed the algebra of Wick powers for nonlinear Sigma models
- We have given a proof of the emergence of Ricci flow

To do

- Compute higher order flows (RG-2 flow)
- Develop pAQFT to its fullest extent for Euclidean theories
- Apply it for all physically relevant models out there ....
Conclusions

And now?

Outlook

- We have developed a general framework to discuss Euclidean AQFT
- We have constructed the algebra of Wick powers for nonlinear Sigma models
- We have given a proof of the emergence of Ricci flow

To do

- Compute higher order flows (RG-2 flow)
- Develop pAQFT to its fullest extent for Euclidean theories
- Apply it for all physically relevant models out there ....