Classical field models for condensates of light

Michiel Wouters

Bad Honnef, August 11, 2025

TQC
Universiteit Antwerpen

Outline

Lecture 1

- Stochastic classical field models for polariton and photon condensation
- Excitation spectrum and Goldstone mode

Lecture 2

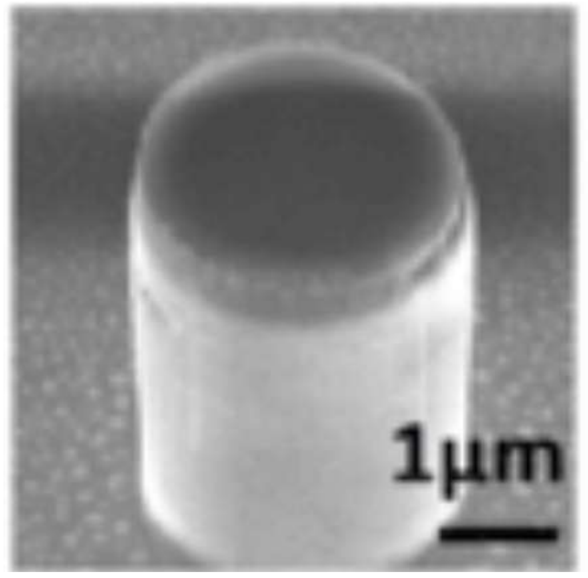
- Scaling properties of the phase fluctuations
- Experimental observation of KPZ scaling

I. Carusotto and C. Ciuti RMP 2013,
I. Carusotto, J. Bloch and MW. Nat. Phys. Rev. 2022

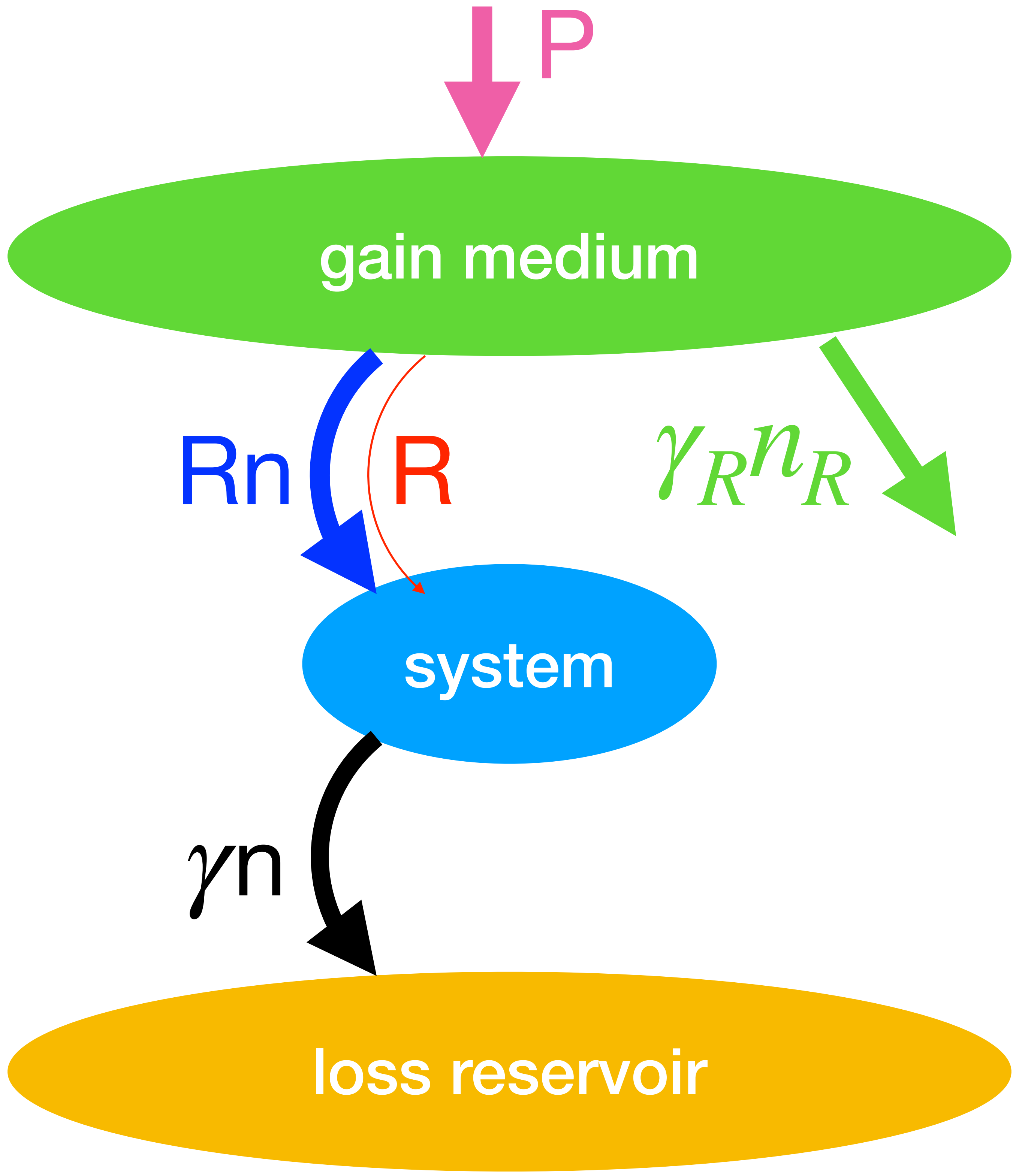
Single mode rate equation

$$\frac{dn}{dt} = R(n + 1) - \gamma n$$

stimulated emission   spontaneous emission



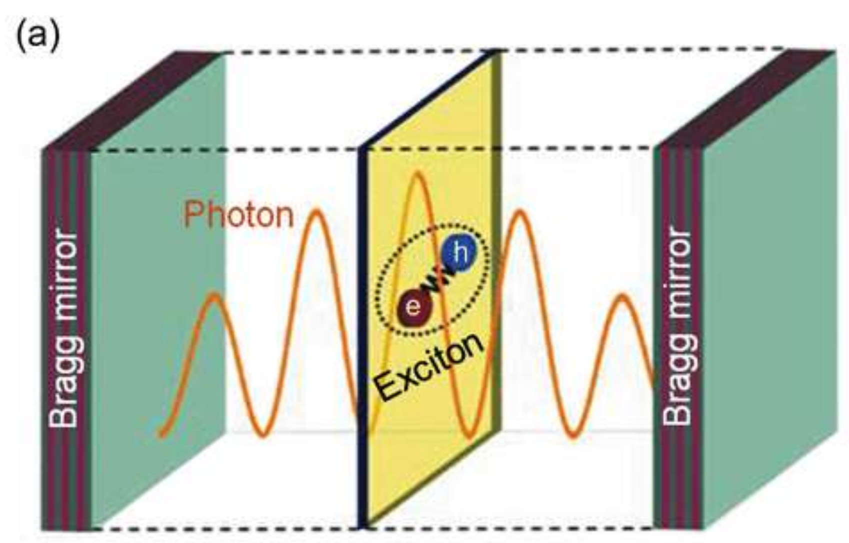
C2N, Paris

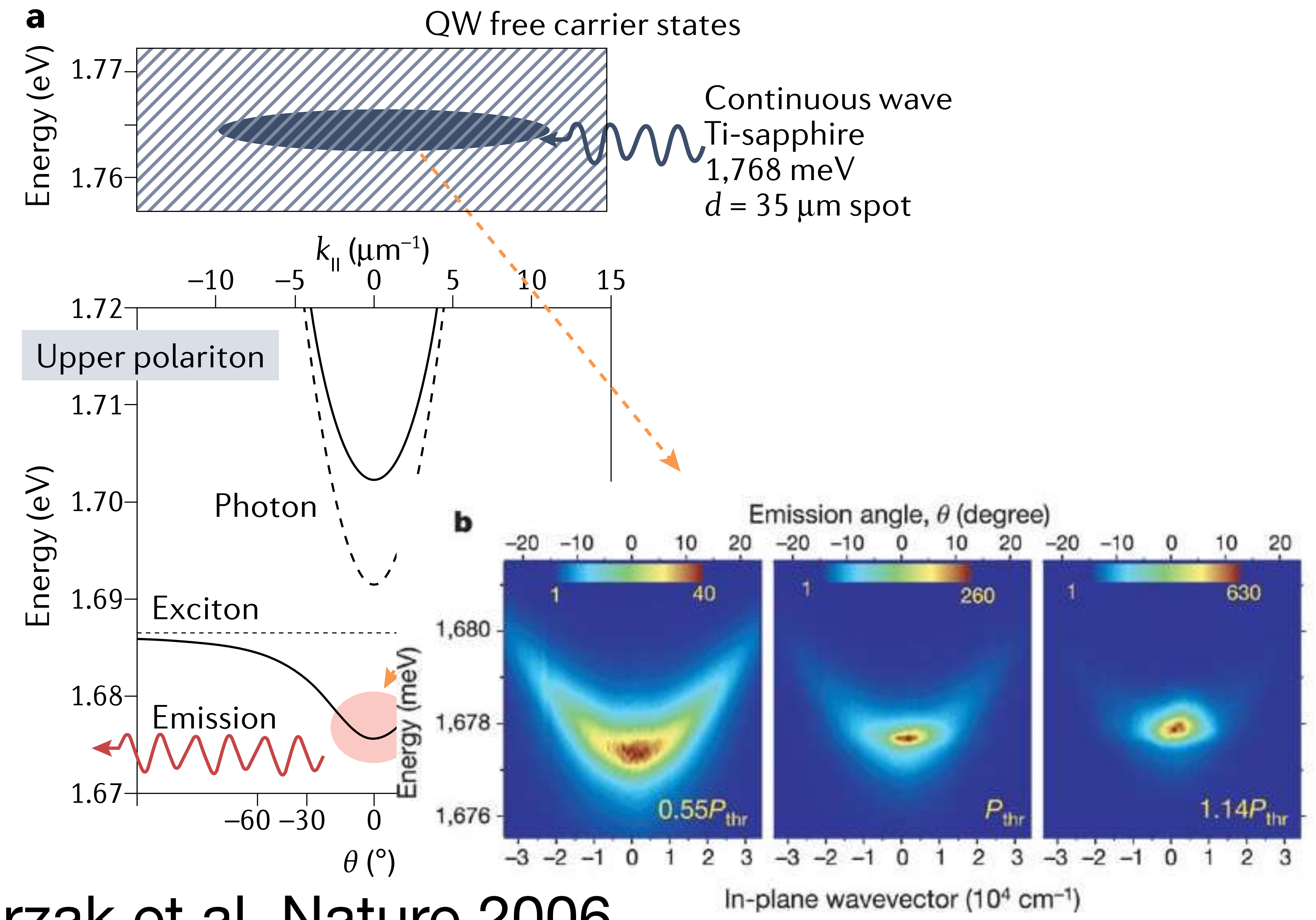


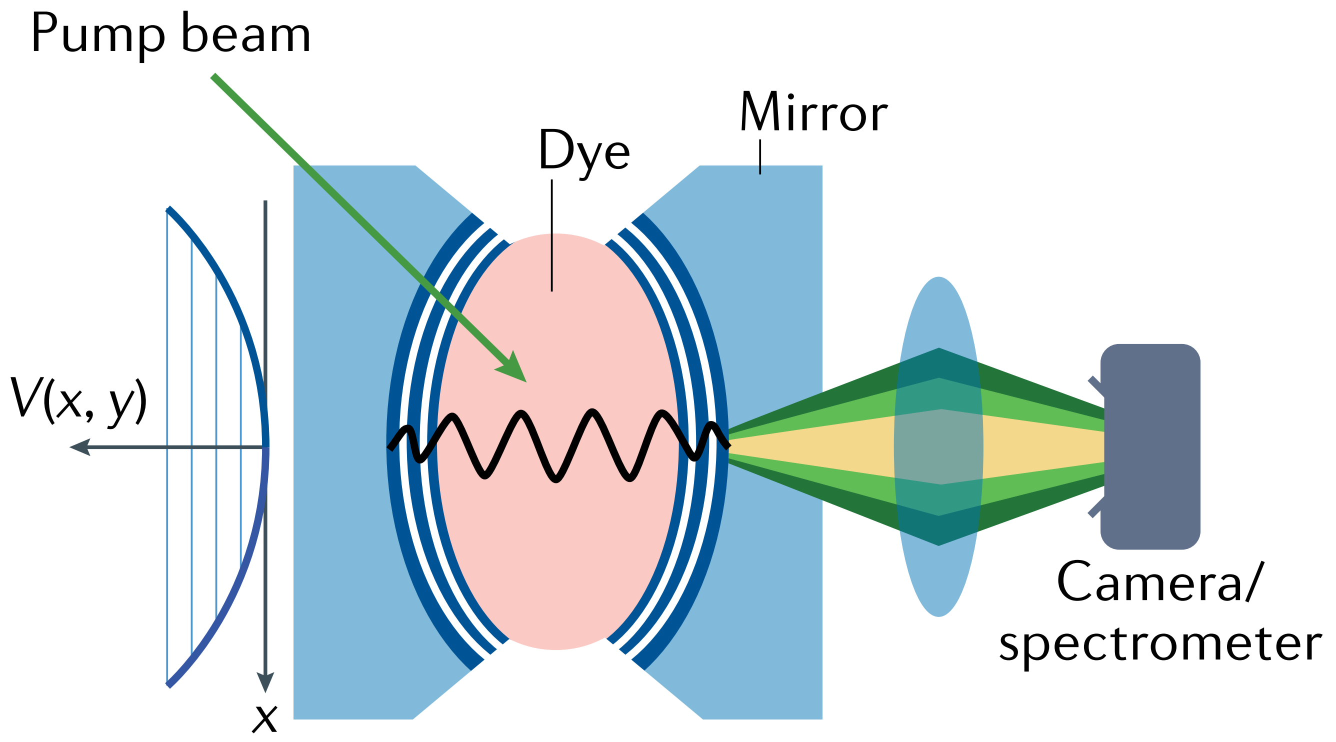
assume constant R

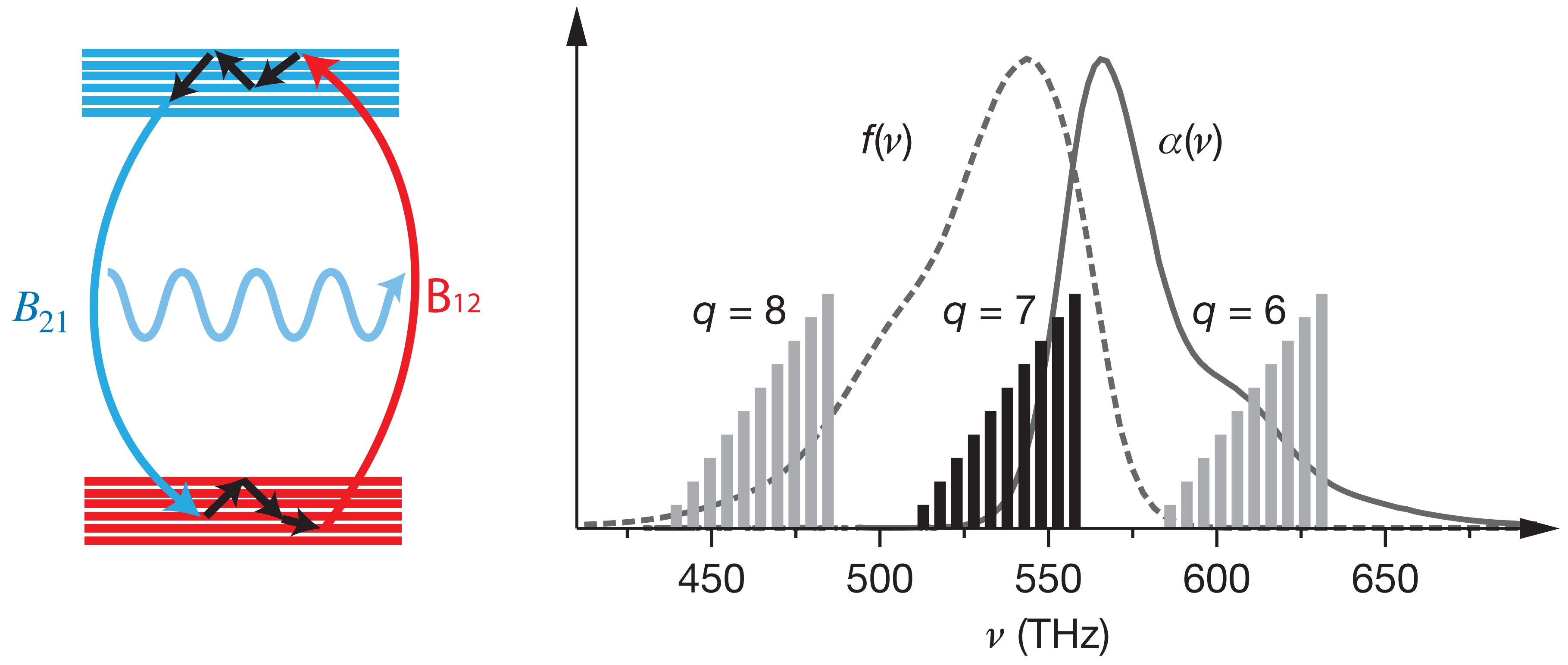
$\Rightarrow$ steady state $n = \dfrac{1}{\frac{\gamma}{R} - 1}$ (requires $\gamma > R$)

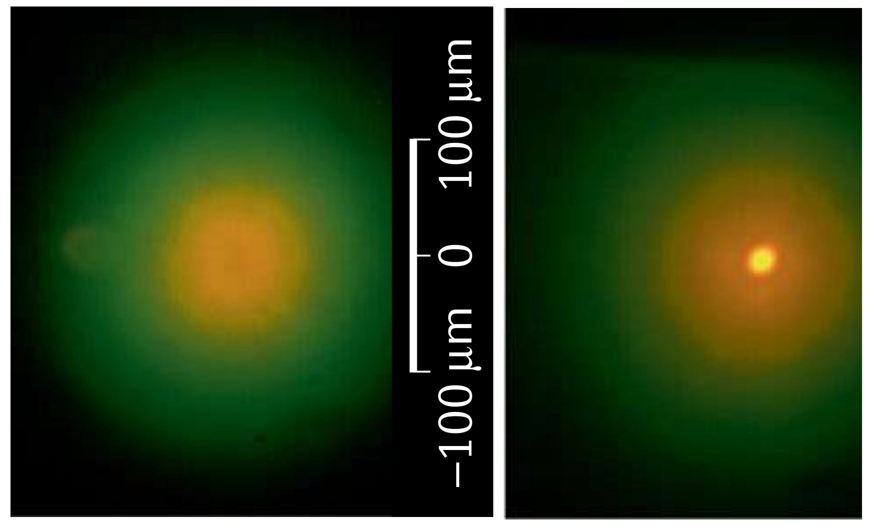
Polariton and photon condensation











Kasprzak et al. Nature 2006

Klaers et al. Nature 2010

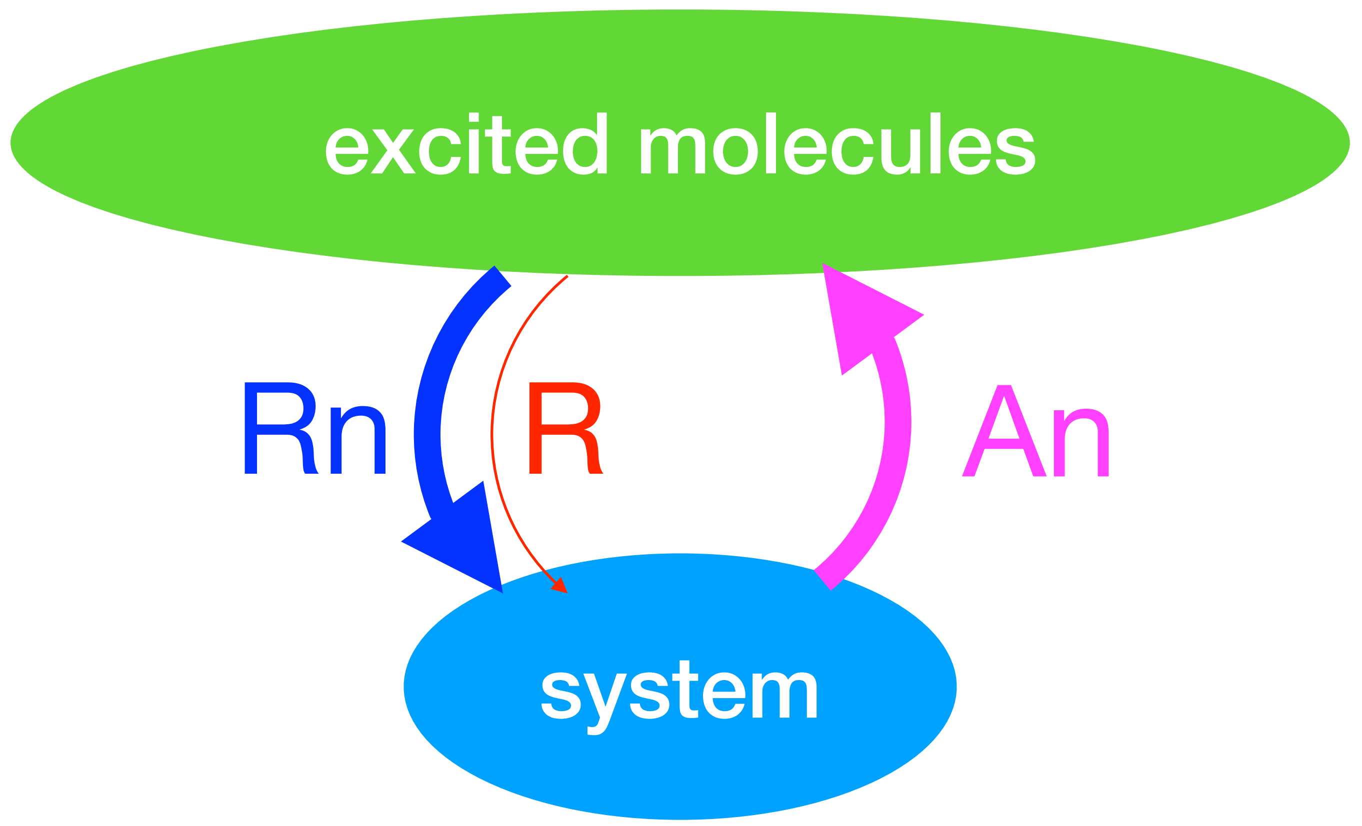
Thermalization (photon condensates)

reabsorption

$$\frac{dn}{dt} = R(n + 1) - An$$

stimulated emission   spontaneous emission

thermal equilibrium



steady state $n = \dfrac{1}{\frac{A}{R} - 1}$ (requires $A > R$)

$$\frac{A}{R} = \exp\left(\frac{\epsilon - \mu}{k_B T}\right)$$

detailed balance
Kennard-Stepanov
Van Roosbroek-Shockley
Kubo-Martin-Schwinger

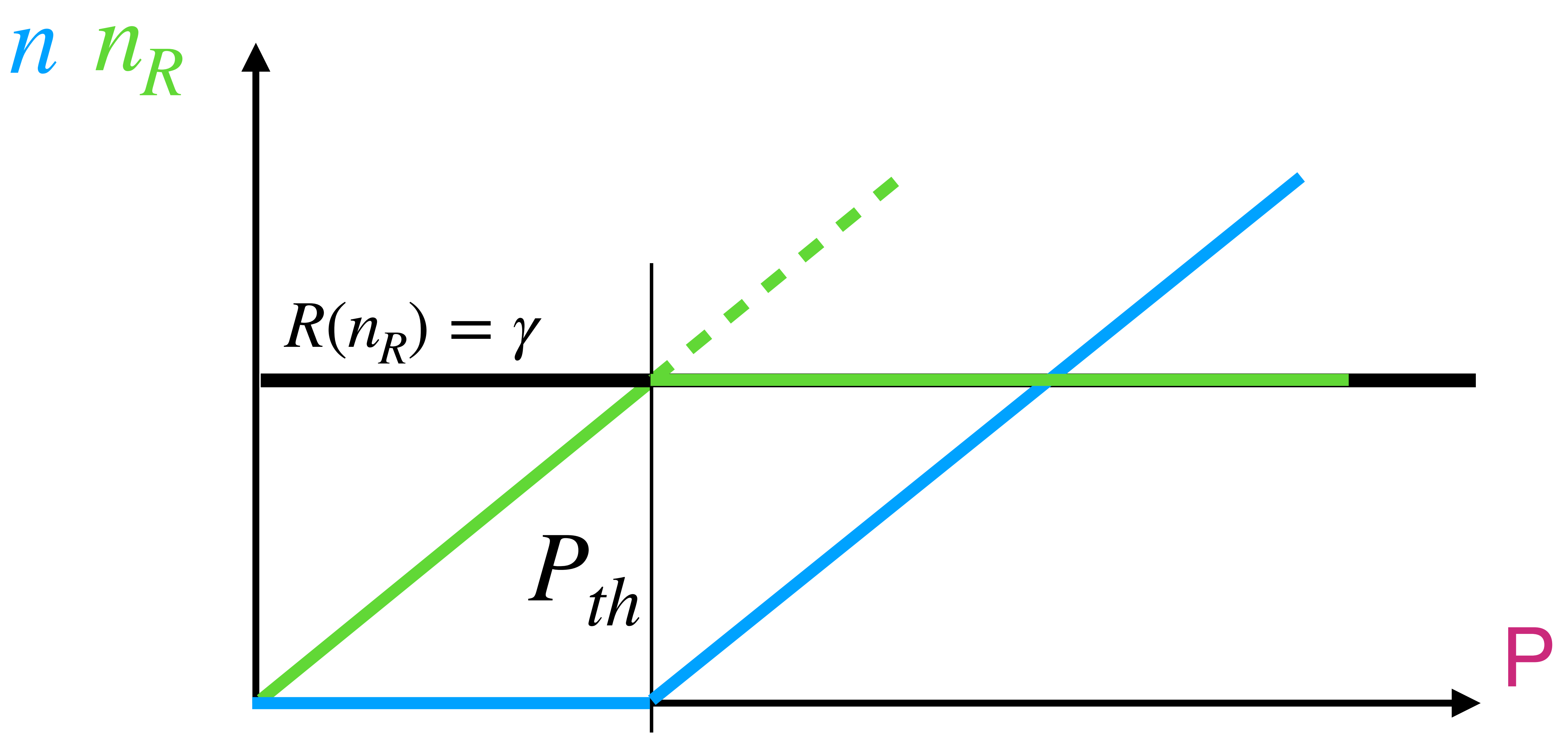
$$n = \frac{1}{\exp\left(\frac{\epsilon - \mu}{k_B T}\right) - 1}$$

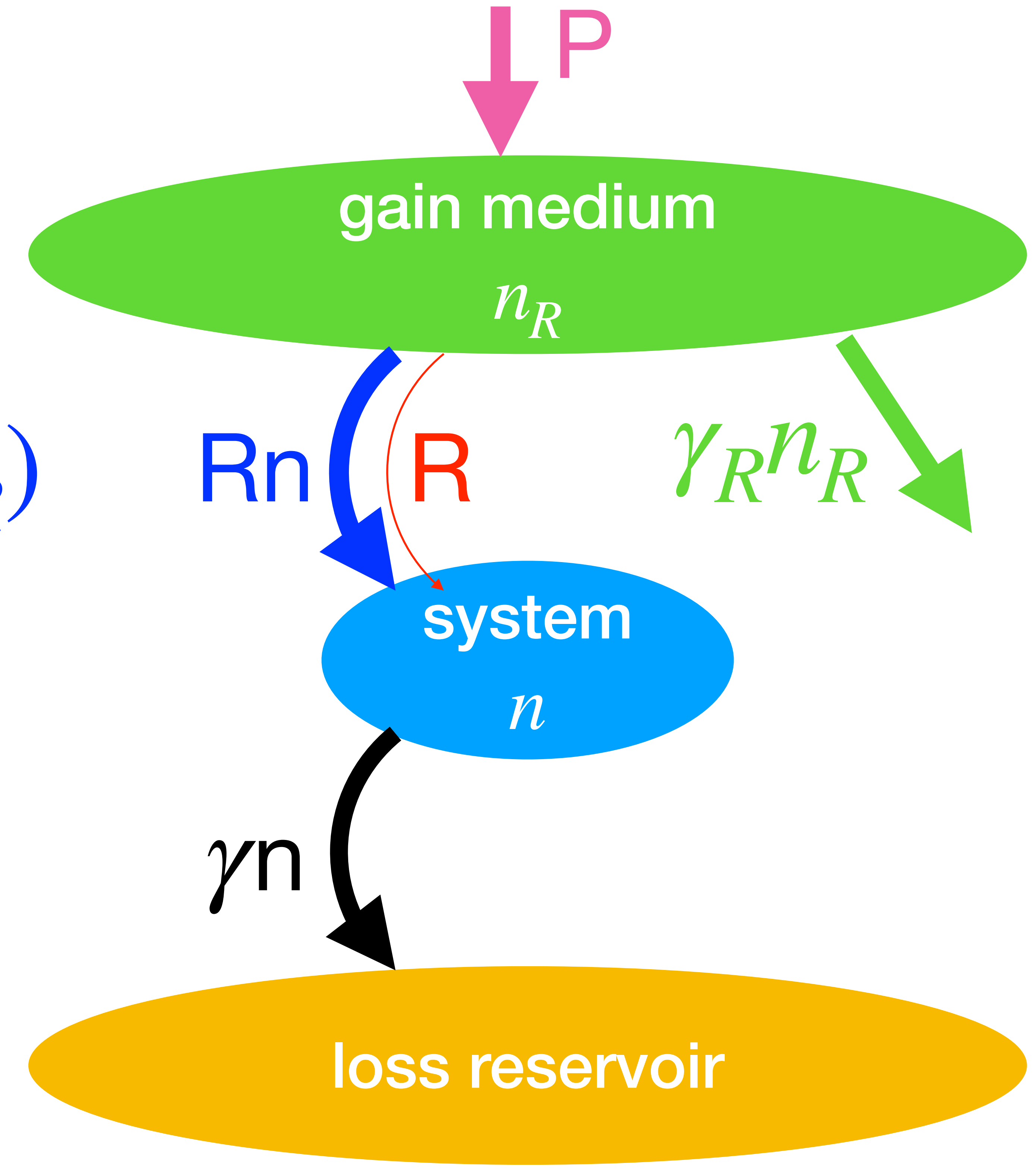
$\Rightarrow$ equilibrium Bose-Einstein distribution

Reservoir dynamics

$$\frac{dn_R}{dt} = -R(n_R)(n + \cancel{1}) - \gamma_R n_R + P$$

$$\frac{dn}{dt} = R(n_R)(n + \cancel{1}) - \gamma n$$





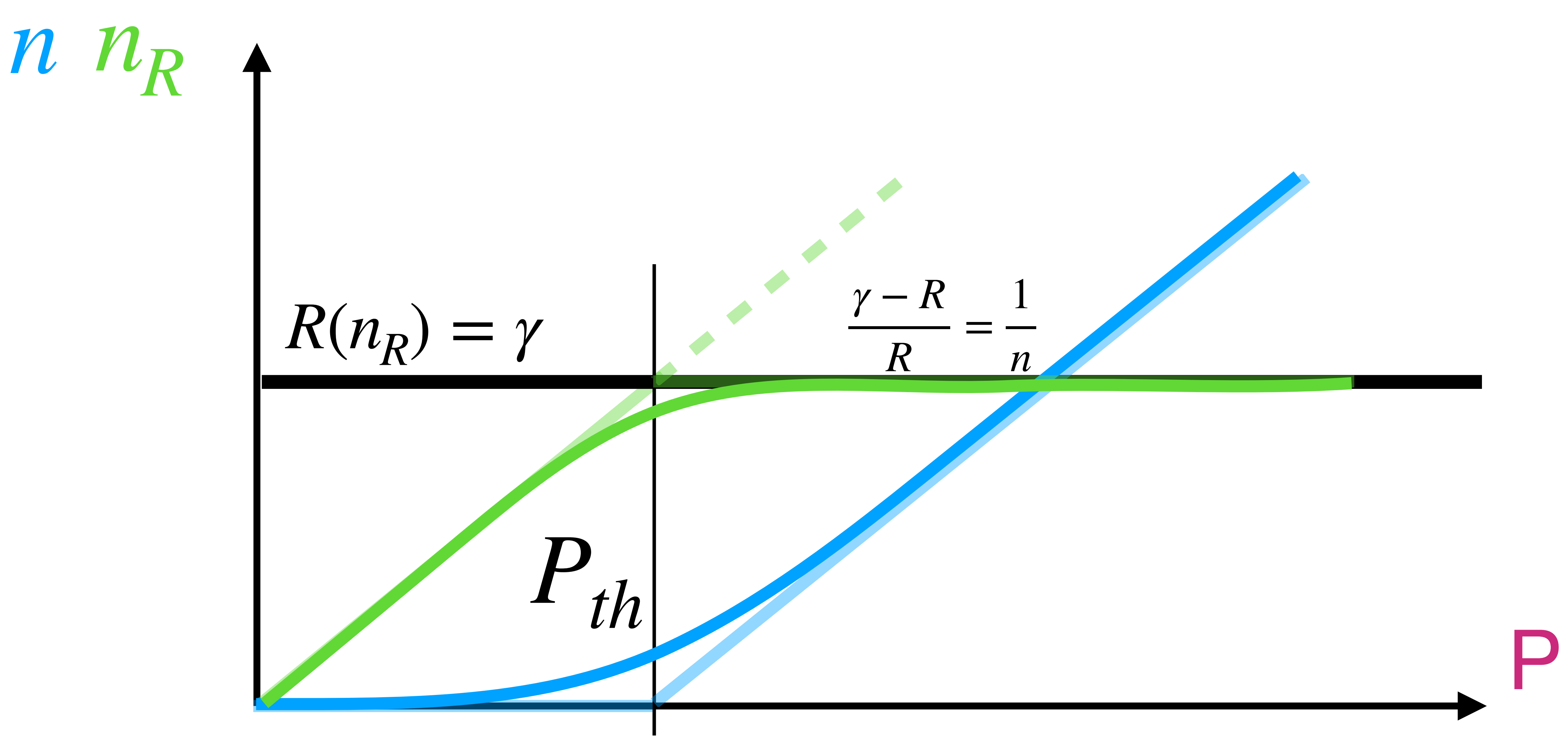
$R \equiv R(n_R)$

$\gamma_R \to 0$: thresholdless

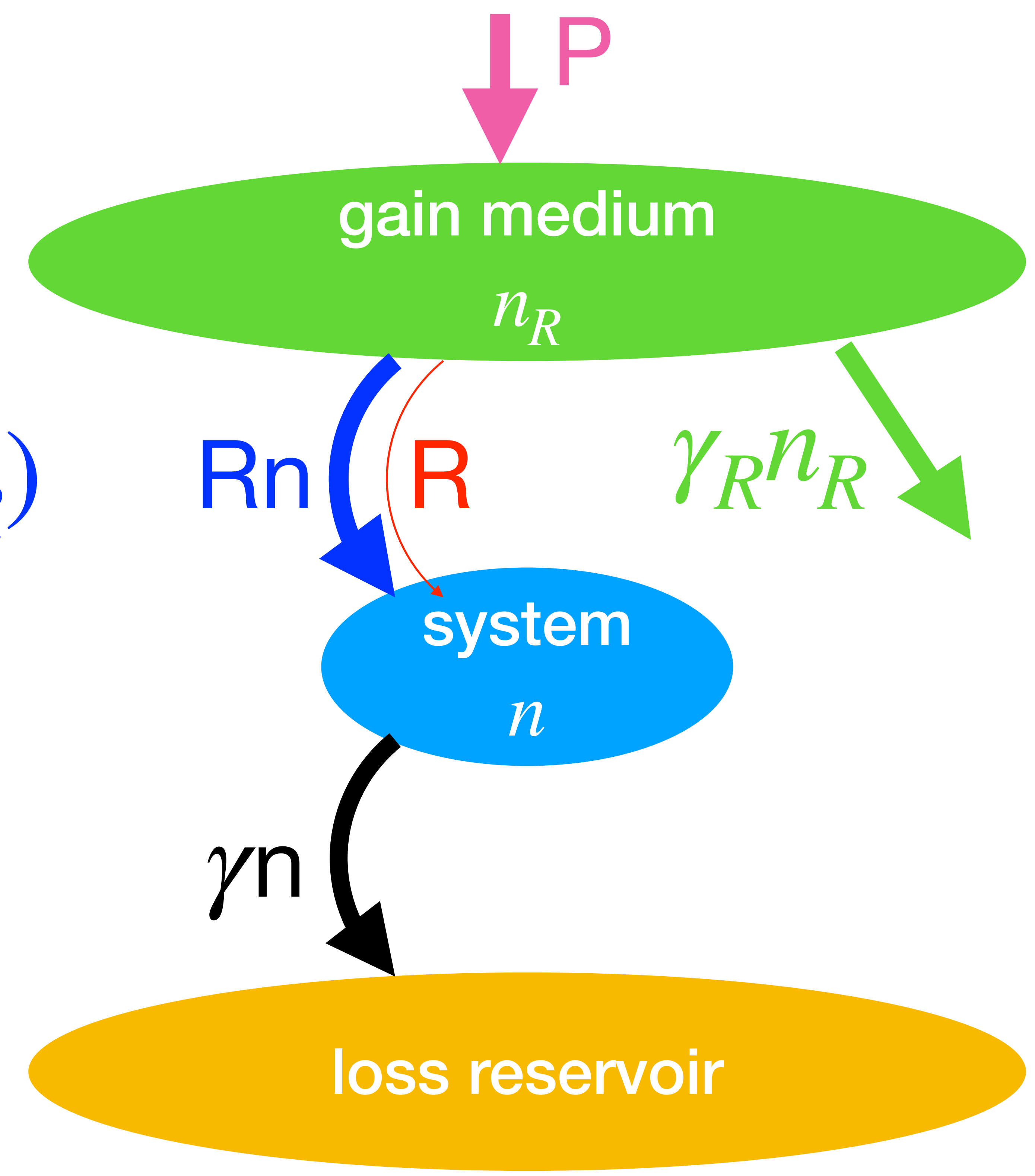
Steady state (with sp. em.)

$$\frac{dn_R}{dt} = -R(n_R)(n+1) - \gamma_R n_R + P$$

$$\frac{dn}{dt} = R(n_R)(n+1) - \gamma n$$



$R \equiv R(n_R)$



$\gamma_R \to 0$: thresholdless

In terms of amplitude

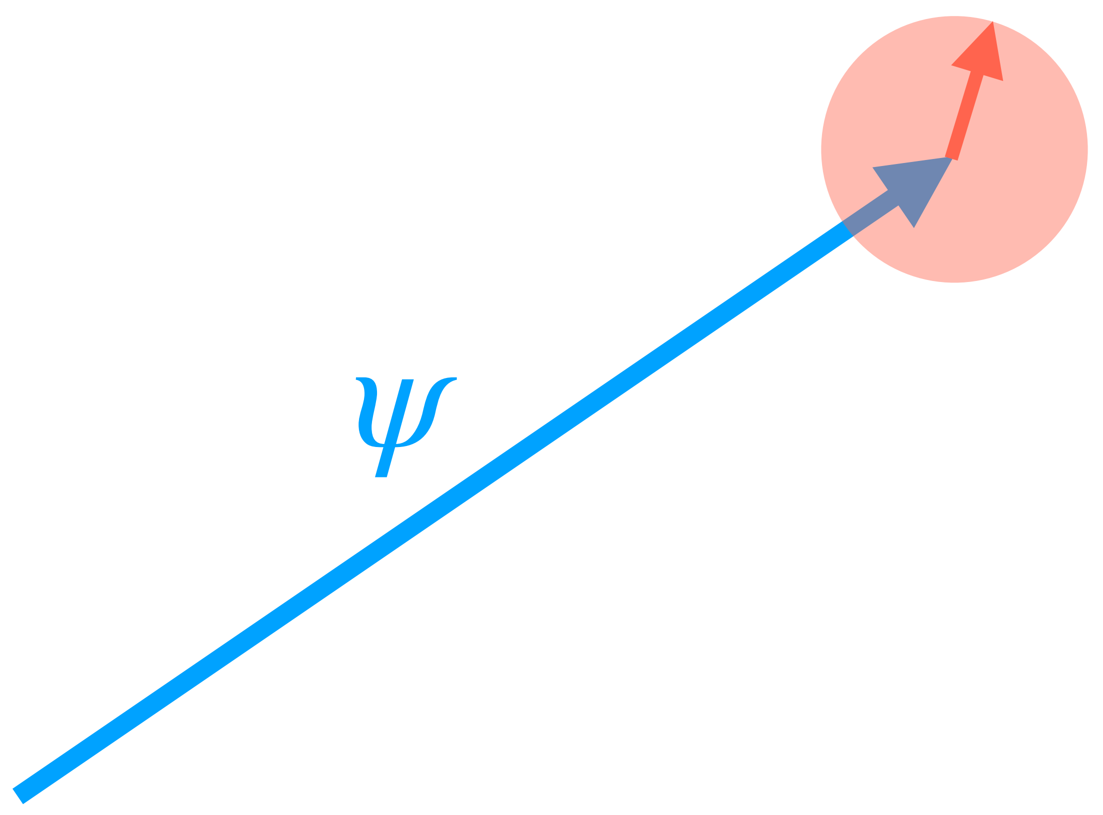
$$\psi = \sqrt{n}\; e^{i\theta}$$

$$\frac{d\psi}{dt} = \frac{1}{2}(R - \gamma)\psi \qquad \Rightarrow \qquad \frac{dn}{dt} = R(n + \cancel{1}) - \gamma n$$

spontaneous emission is a quantum effect

Quantum fluctuations can be introduced in a first approximation through stochastic term in the amplitude equation.

Henry phasor model



add unit phasor at rate of spont. em. (R)

$\sim$ Wiener noise when avg. over times $\gg 1/R$

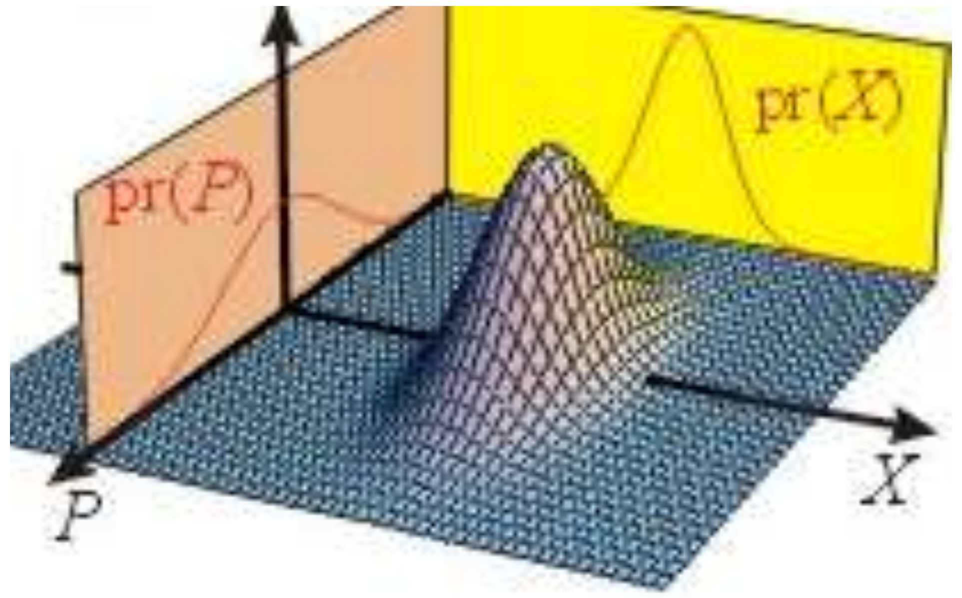
$d\psi = \ldots + \sqrt{R/2}\; dW$ with $\langle dW^* dW \rangle = 2\, dt$

$$d\langle |\psi|^2 \rangle = R(\langle |\psi|^2 \rangle + 1) - \gamma \langle |\psi|^2 \rangle$$

C.Henry IEEE **18, 259** (1982)

Wigner formulation

Wigner quasi-probability distribution $P_W(\psi, \psi^*)$



$\psi = (x + ip)/\sqrt{2}$

Gives symmetrised quantum expectation values, e.g.

$$\langle \psi^* \psi \rangle_W = \frac{1}{2} \langle \hat{\psi}^\dagger \hat{\psi} + \psi \hat{\psi}^\dagger \rangle = \langle \hat{\psi}^\dagger \hat{\psi} \rangle + \frac{1}{2}$$

→ 1/2 quantum fluct. per mode

From operator correspondences (truncation when interacting)

$$d\psi = \frac{1}{2}(R - \gamma)\psi + \sqrt{\frac{R + \gamma}{4}}\, dW$$

cf. Henry $d\psi = \ldots + \sqrt{R/2}\, dW$

Phase diffusion (Shawlow-Townes)

$$d\psi = [R(|\psi|^2) - \gamma]\psi\, dt + \sqrt{R/2}\; dW \qquad \psi = \sqrt{n}\; e^{i\theta}$$

above threshold: $d\theta = \sqrt{\dfrac{R}{2n}}\, dW_\theta$ with $\langle dW_\theta dW_\theta \rangle = dt$

no restoring force because of spontaneous symmetry breaking

phase diffusion: $\langle [\theta(t) - \theta(0)]^2 \rangle = \dfrac{R}{2n}\, t$

first order coherence: $\langle \psi^*(t)\psi(0) \rangle \approx n\langle e^{-i[\theta(t)-\theta(0)]} \rangle = ne^{-\frac{1}{2}\langle [\theta(t)-\theta(0)]^2 \rangle} = e^{-\frac{R}{4n}|t|}$

Density fluctuations

- R constant $\rightarrow g^{(2)} = \dfrac{\langle a^\dagger a^\dagger a a \rangle}{n^2} = 2$.

$$d\psi = [R(|\psi|^2) - \gamma]\psi\, dt + \sqrt{R/2}\, dW$$

- Gain saturation reduces density fluctuations

- Expand: $R(n_0 + \delta n) \approx R_0 - R_1 \delta n$,

$$g^{(2)} = 1 + \frac{1}{1 + \frac{n_0^2}{M_{\text{eff}}}}$$ with $M_{\text{eff}} = \dfrac{R_0}{R_1}$

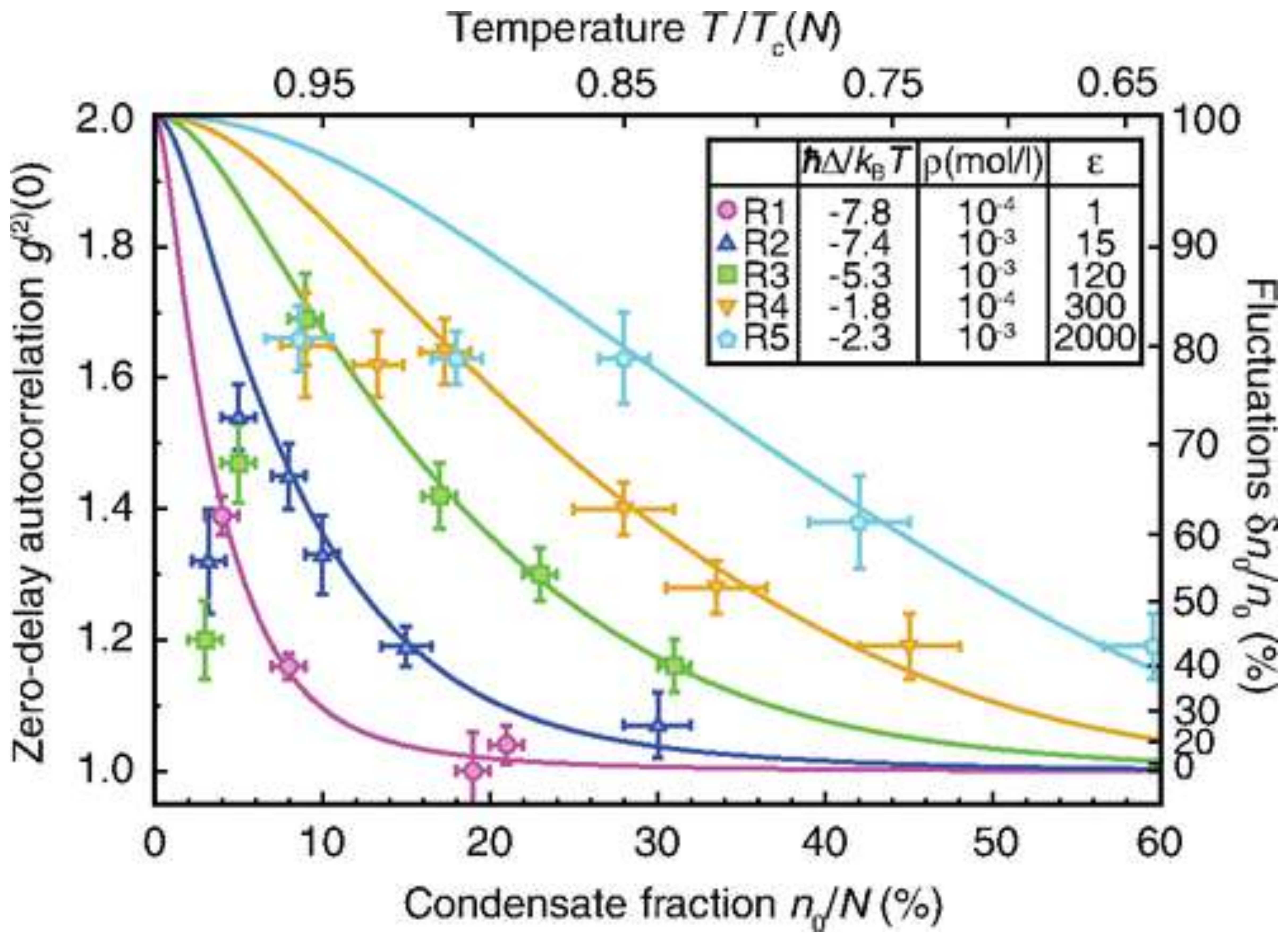
- Quasi-condensate: $\dfrac{n_{qc}}{n} = \sqrt{2 - g^{(2)}} : \psi = n_{qc} e^{i\theta} + \psi'$

Svistunov, Babaev, Prokof'ev, Superfluid states of matter

density fluctuations in phot. BEC

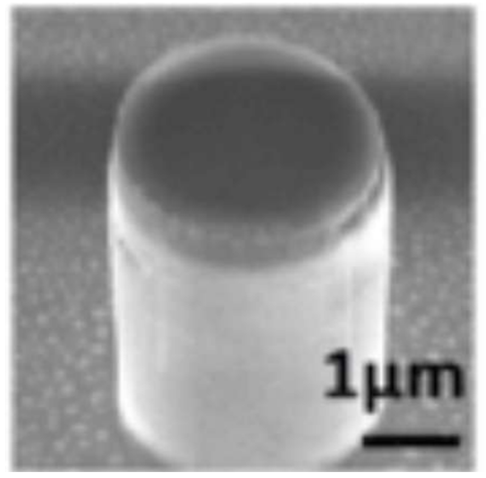
$$g^{(2)} = 1 + \frac{1}{1 + \frac{n_0^2}{M_{\text{eff}}}}$$

for phot. BEC: $M_{\text{eff}} \approx M_{\text{exc}}$



J. Schmitt et al. Phys. Rev. Lett. 112, 030401 (2012)

Spatially extended systems



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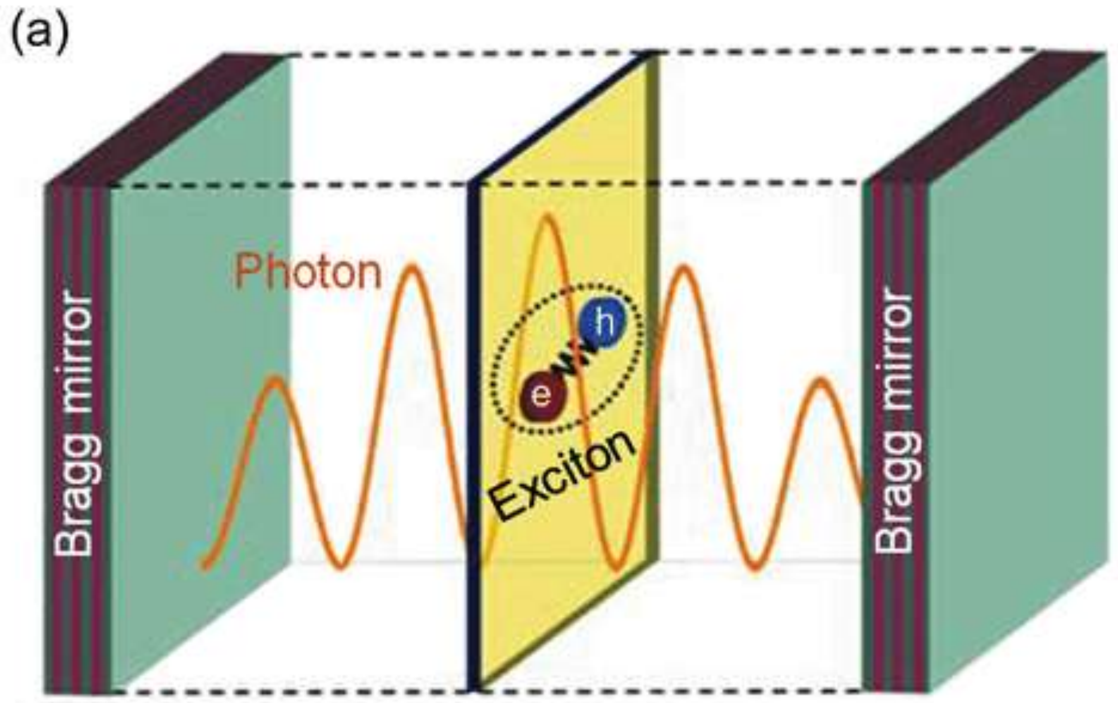
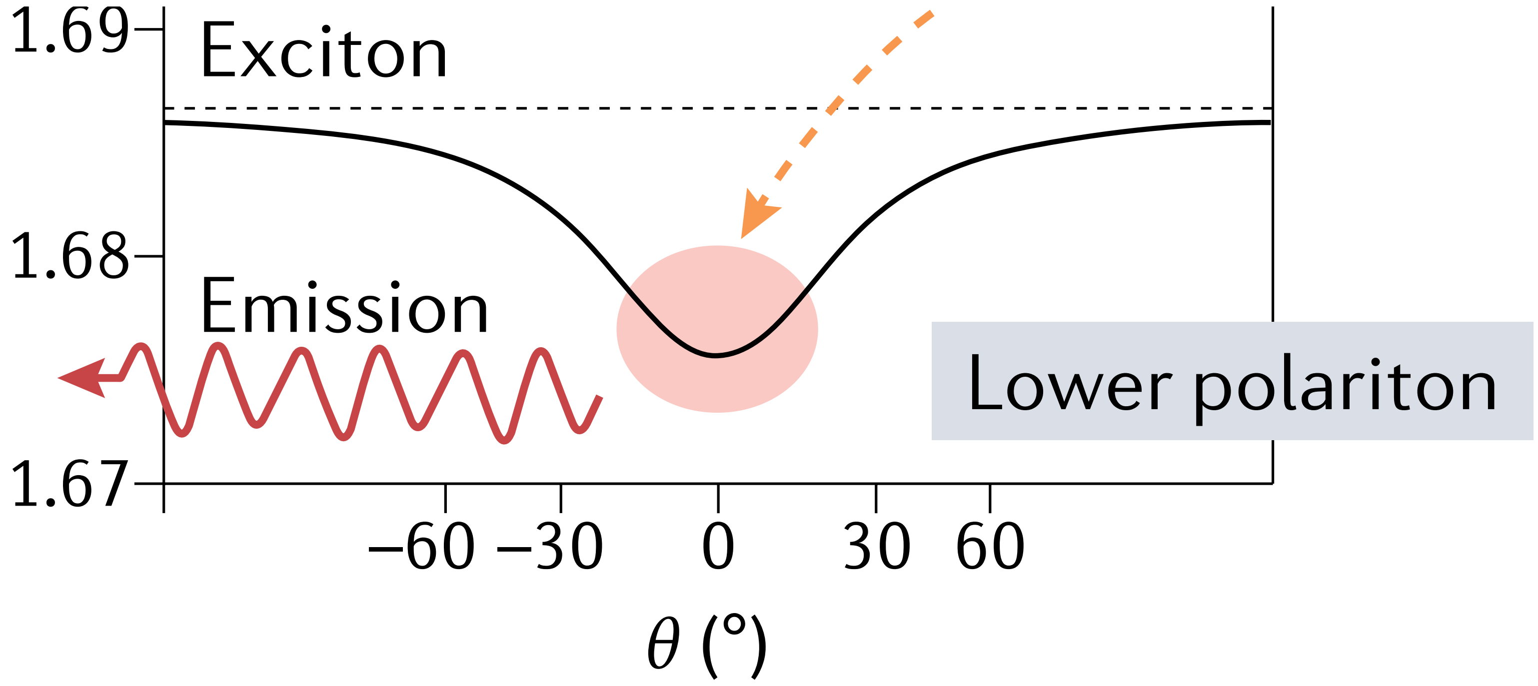


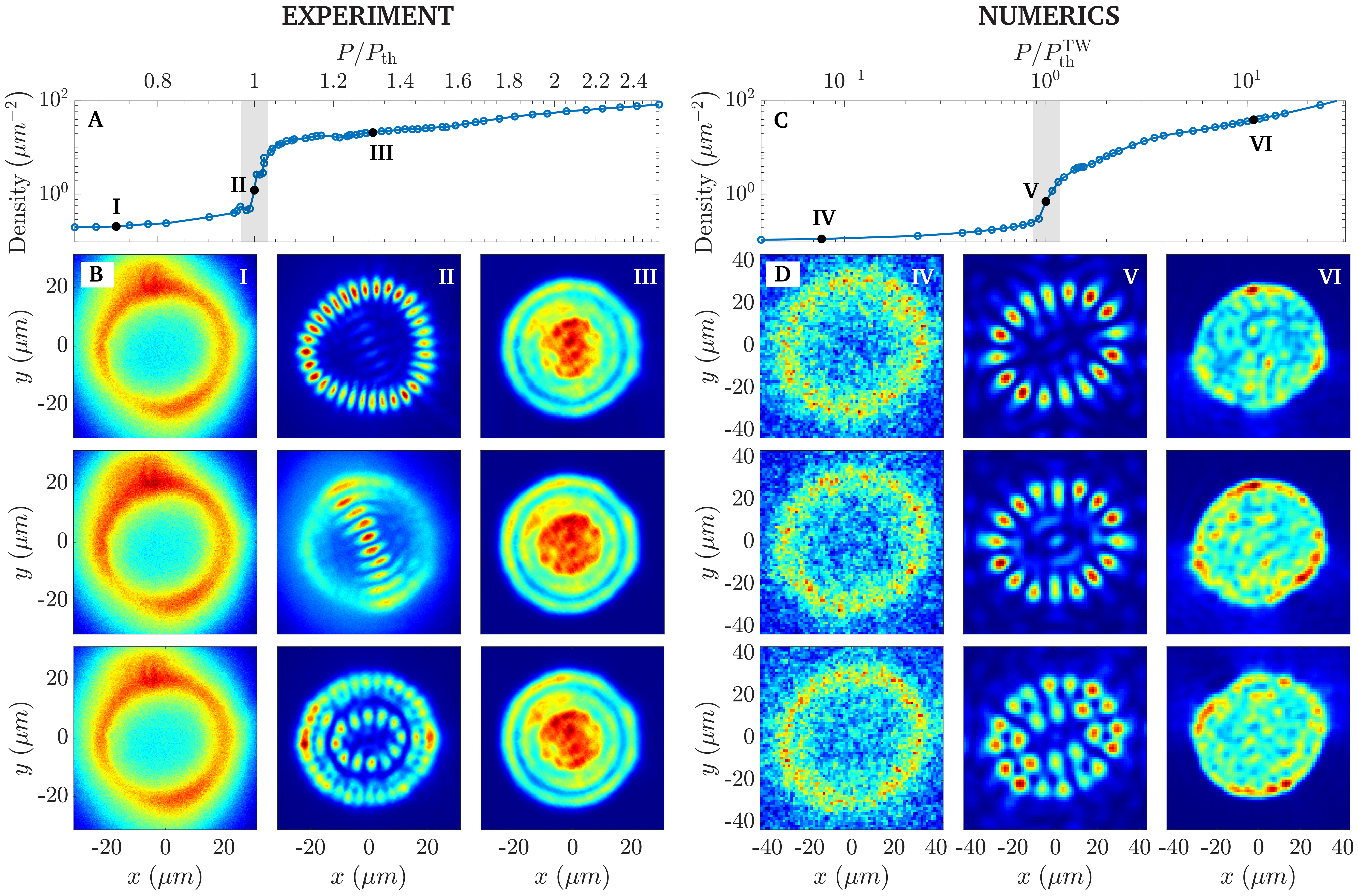


$$i\frac{\partial}{\partial t}\psi(x,t) = -\frac{\nabla^2}{2m}\psi(x,t) + V(x)\,\psi(x,t) + g\,|\psi(x,t)|^2 + g_R n_R(x,t)\,\psi(x,t)$$
$$+\frac{i}{2}\{R[n_R(x,t)] - \gamma\}\,\psi(x,t) + \sqrt{\frac{R+\gamma}{4\Delta x}}\xi(x,t)$$

Shaping Hermitian and non-Hermitian terms → non-Hermitian physics, e.g. exeptional points

E. Estrecho et al. Nature **526**, 554 (2020)

Simulations vs experiment



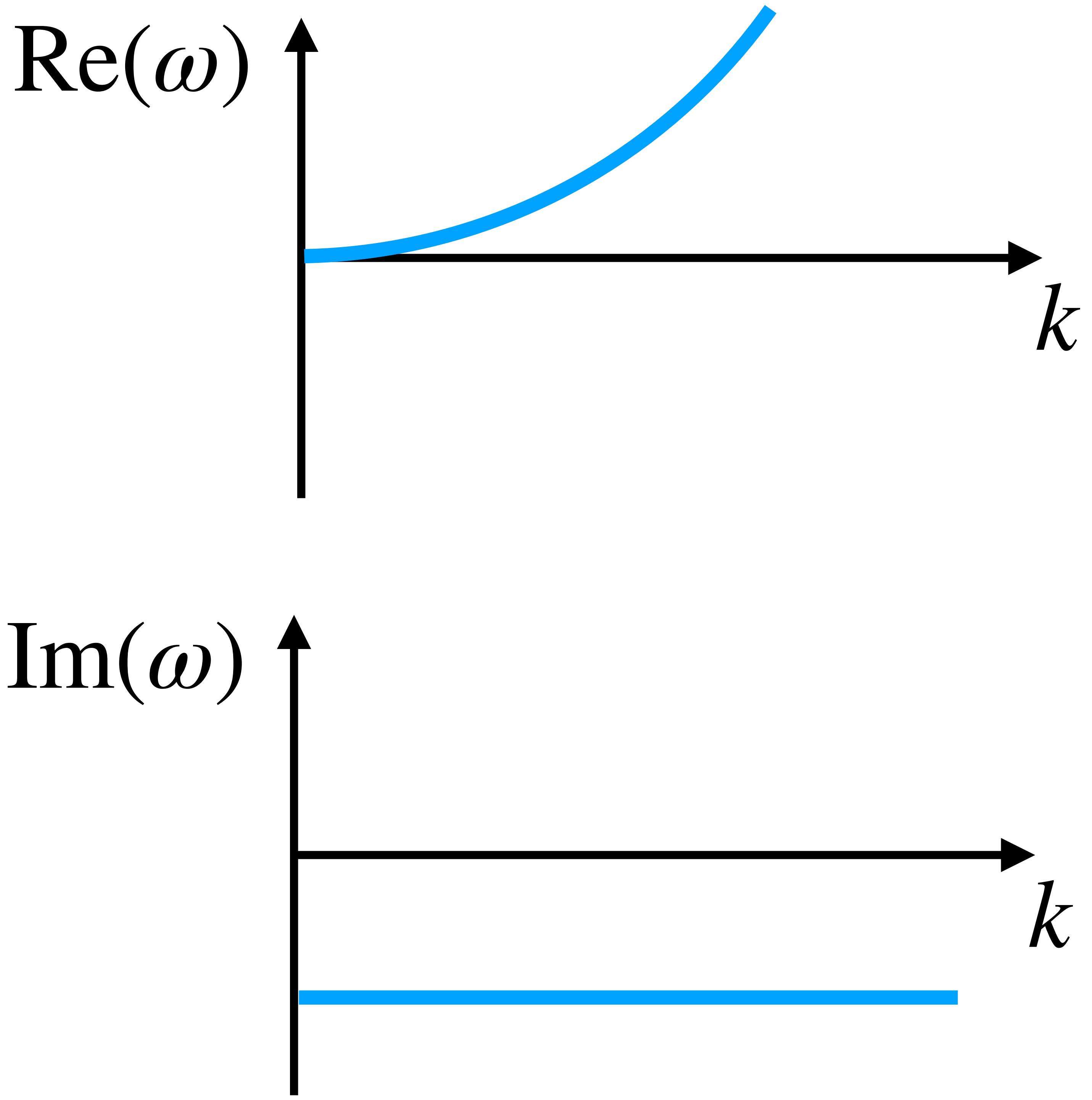
H. Alnatah et al. Science advances **10**, 6762 (2024)

Bogoliubov excitation spectrum

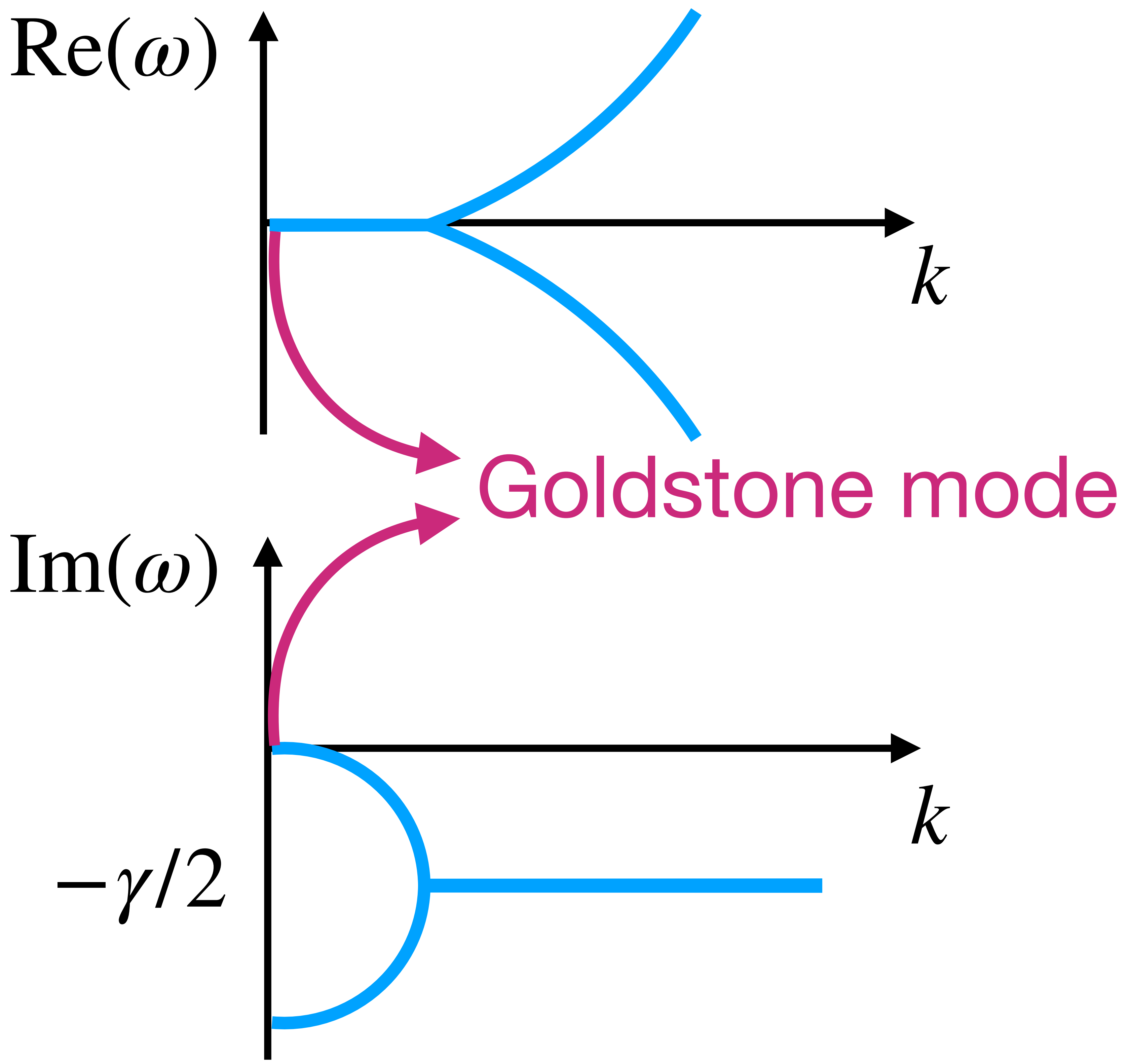
$$\psi = e^{-i\omega_0 t}\left(\psi_0 + u_k e^{-i\omega_k t + ikx} + v_k e^{i\omega t - ikx}\right)$$

linearized eqs. of mot. in u_k and v_k

Empty cavity



Above threshold



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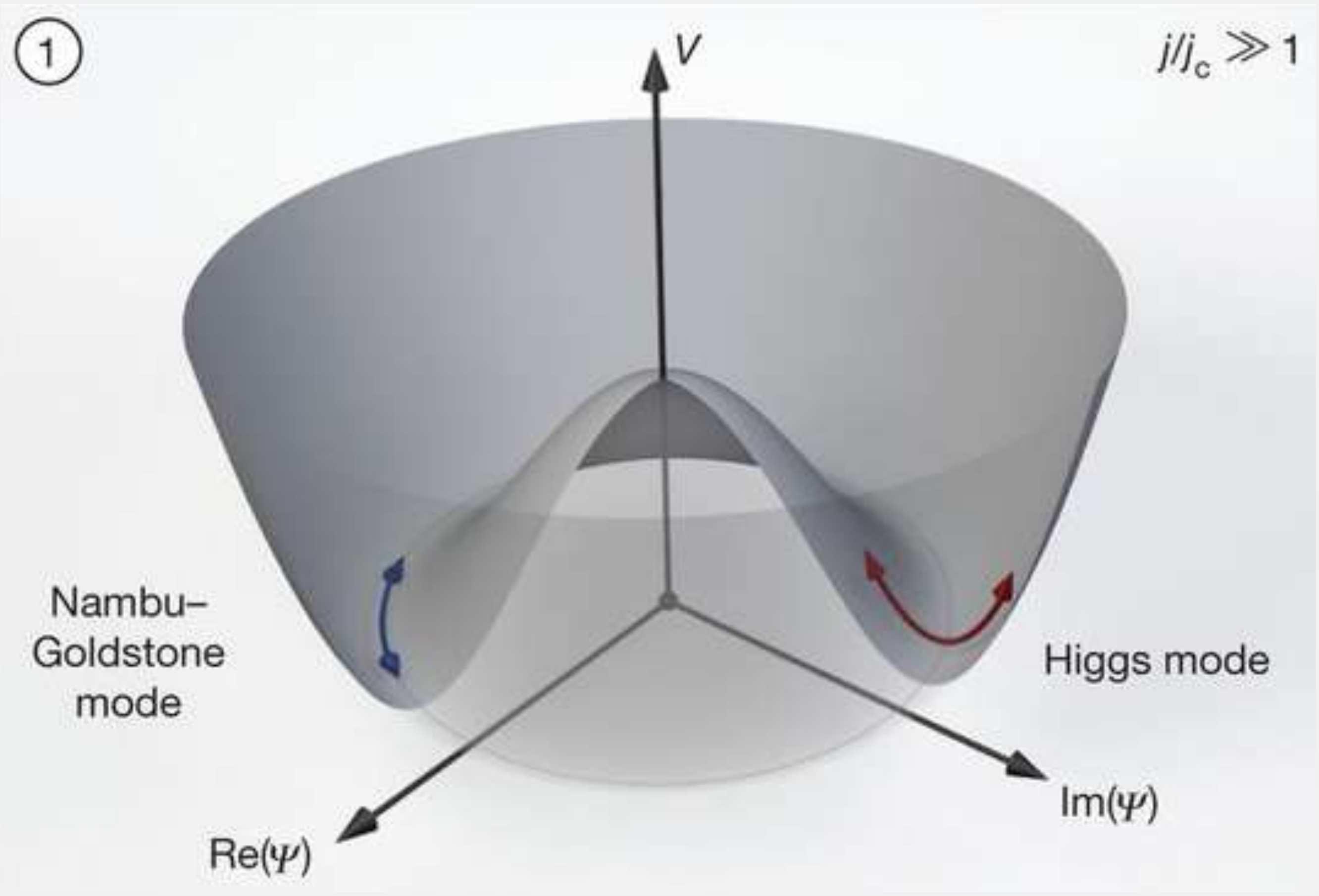
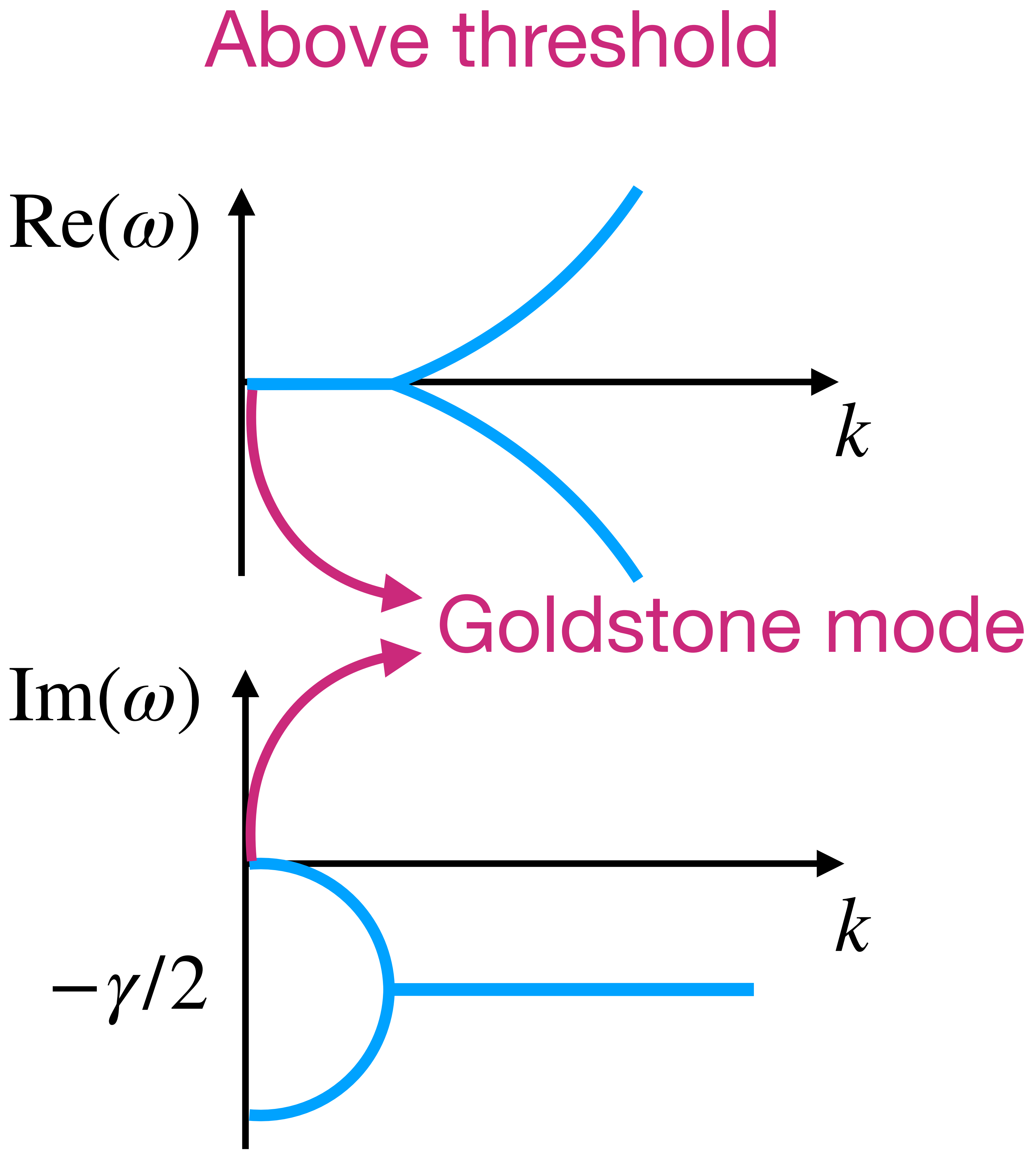
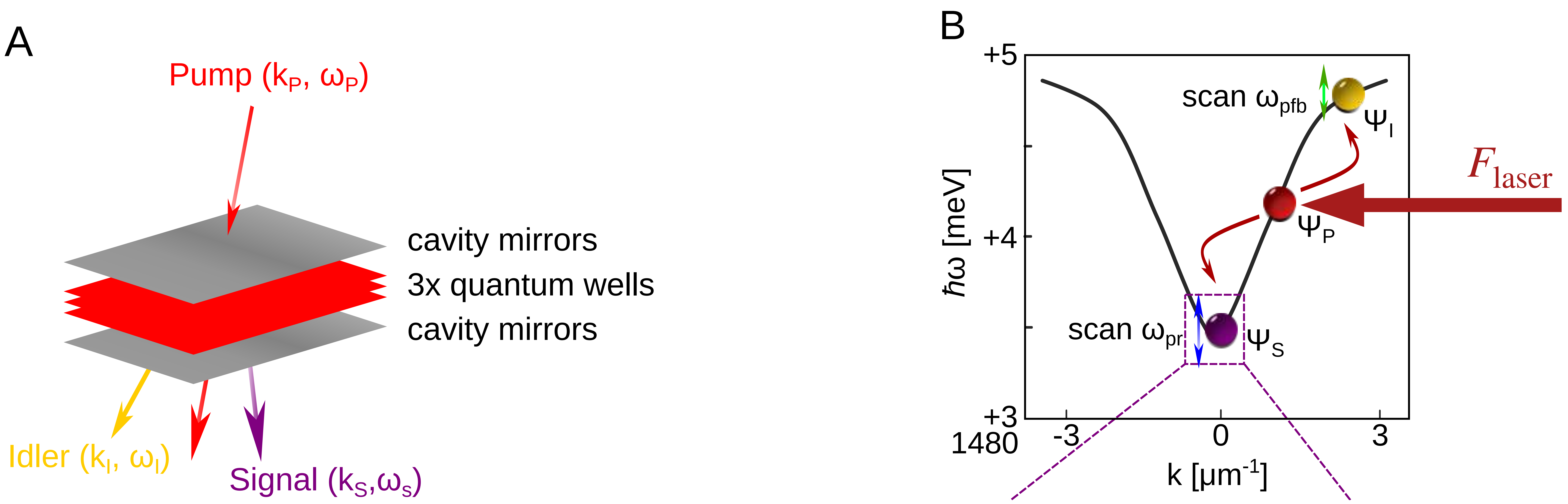


$$i\frac{\partial}{\partial t}\psi(x,t) = -\frac{\nabla^2}{2m}\psi(x,t) + g\,|\psi(x,t)|^2 + g_R n_R(x,t)\,\psi(x,t)$$

$$+\frac{i}{2}\{R[n_R(x,t)] - \gamma\}\,\psi(x,t) + \sqrt{\frac{R+\gamma}{4\Delta x}}\xi(x,t)$$



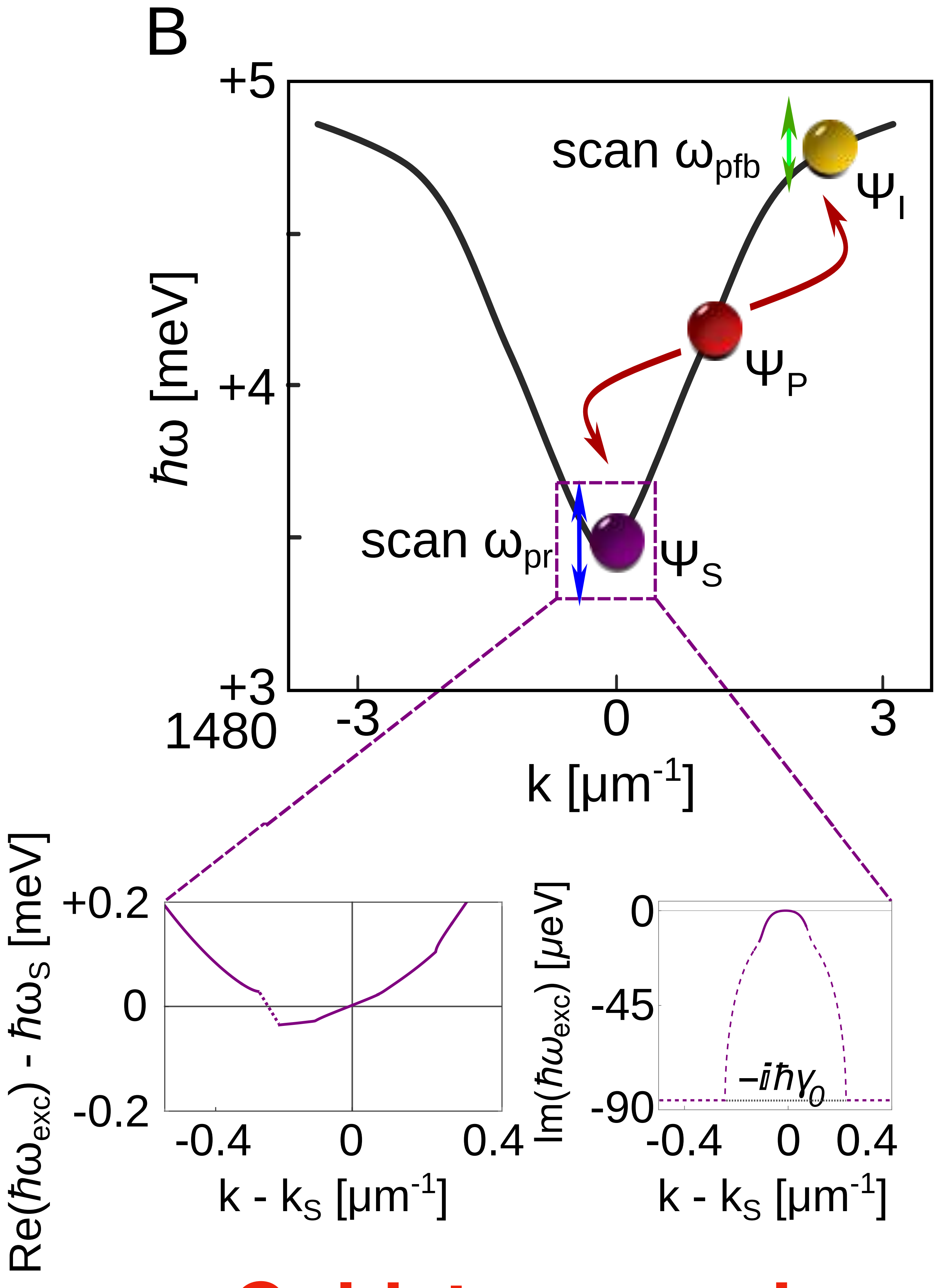
Goldstone mode in polariton OPO



$$i\hbar\frac{\partial\psi(x)}{\partial t} = -\frac{\hbar^2\nabla^2}{2m}\psi(x) + V(x)\psi(x) + g\,|\psi(x)|^2\psi(x) - \frac{i}{2}\gamma\psi(x) + F_{\text{Laser}}\,e^{ik_L x - i\omega_L t}$$

F. Claude et al. Nature Physics **21**, 924 (2025).

OPO excitation spectrum



Goldstone mode

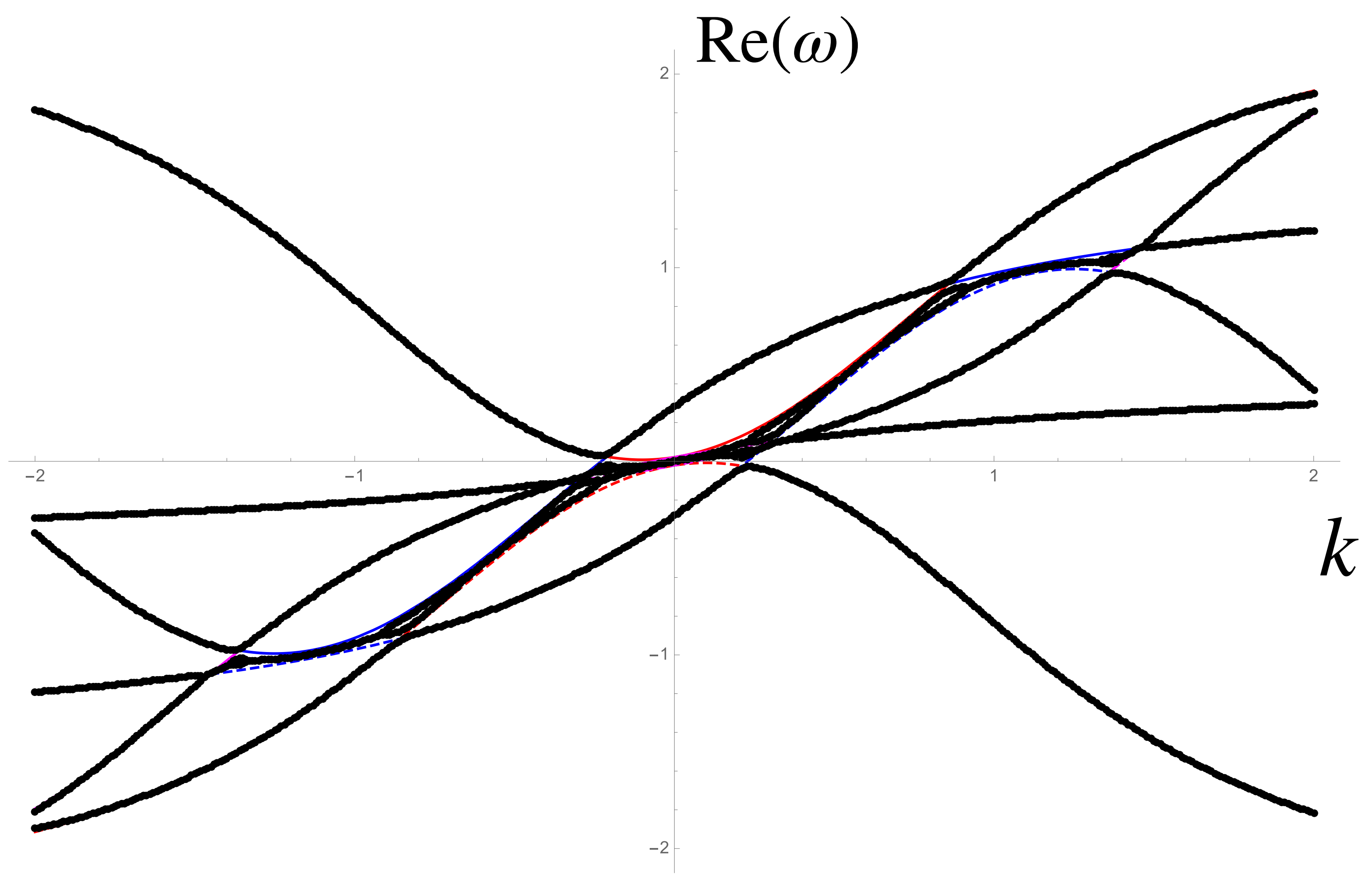
$$i\hbar\frac{\partial\psi(x)}{\partial t} = -\frac{\hbar^2\nabla^2}{2m}\psi(x) + V(x)\psi(x) + g\,|\psi(x)|^2\psi(x) - \frac{i}{2}\gamma\psi(x) + F_{\text{Laser}}\,e^{ik_L x - i\omega_L t}$$

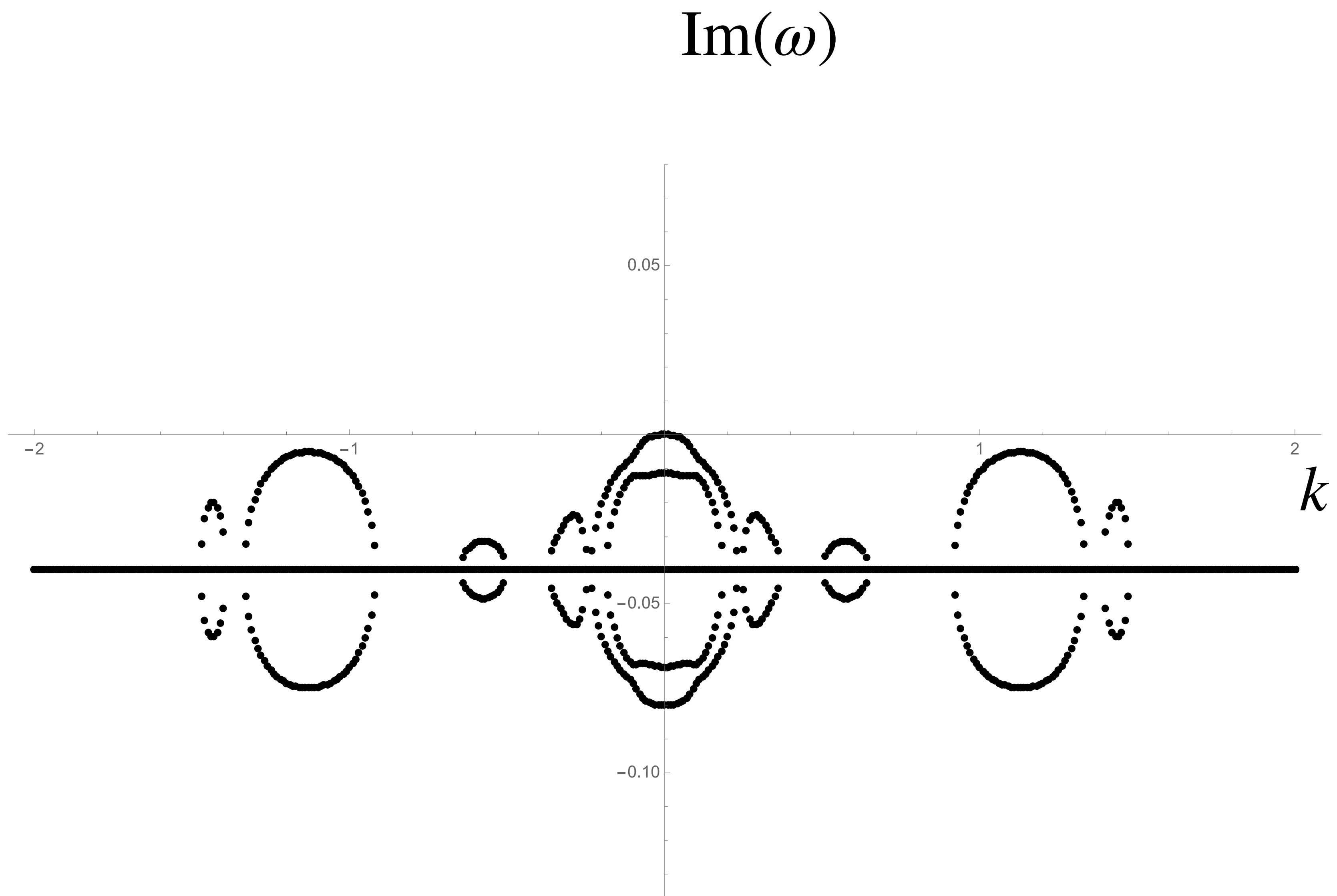
$$\psi(x) = \psi_S e^{ik_S x - i\omega_S t} + \psi_P e^{ik_P x - i\omega_L t} + \psi_I e^{ik_I x - i\omega_I t}$$

U(1) symmetry: $\psi_{s,i} \rightarrow e^{\pm i\theta}\psi_{s,i}$

$$\psi_S = \psi_S^{(0)} + \delta\psi_S(x,t), \quad \psi_P = \psi_P^{(0)} + \delta\psi_P(x,t), \quad \psi_I = \psi_P^{(0)} + \delta\psi_I(x,t)$$

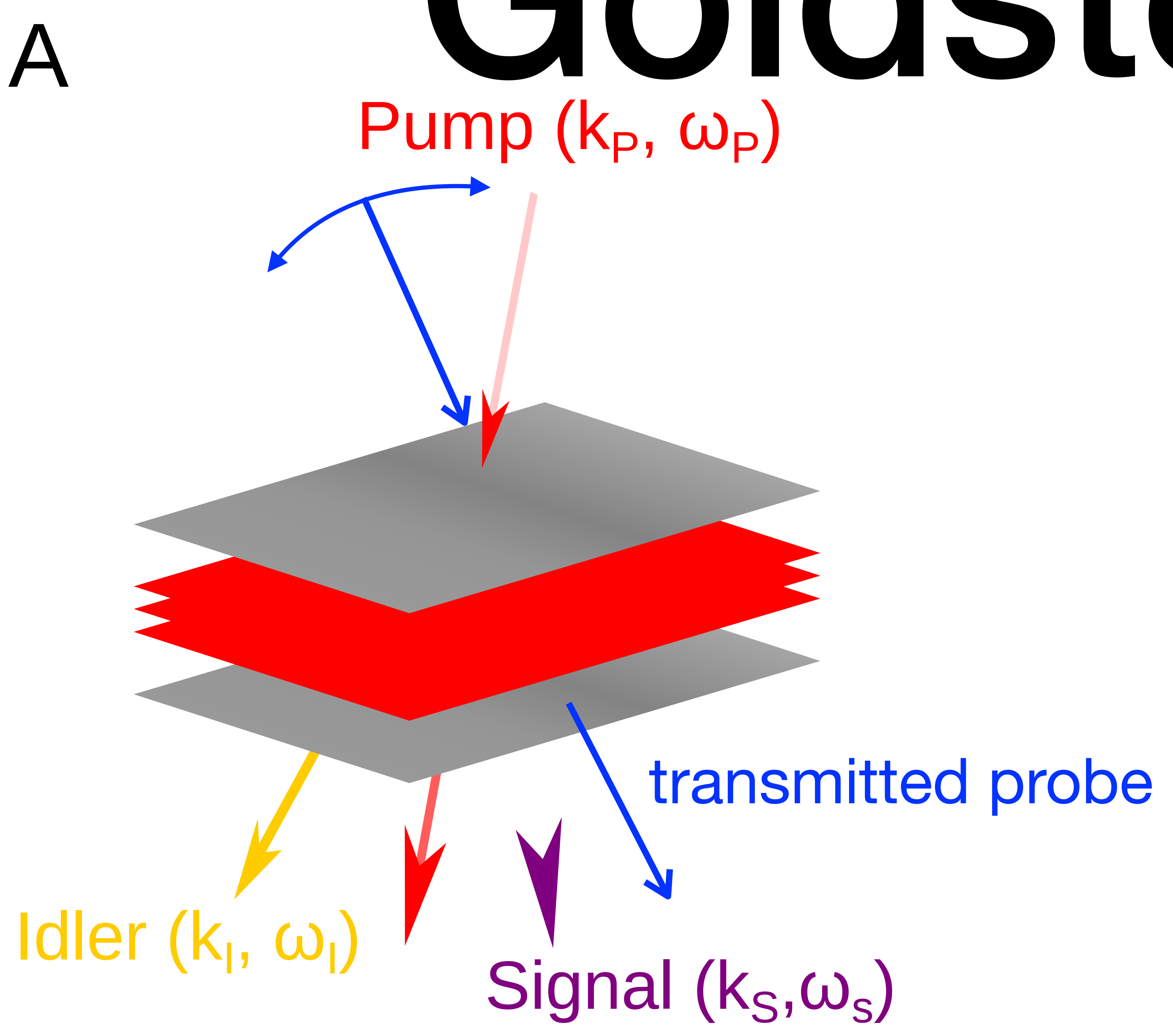
$\delta\psi_{S,P,I}$ are complex $\Rightarrow$ 6 excitation branches



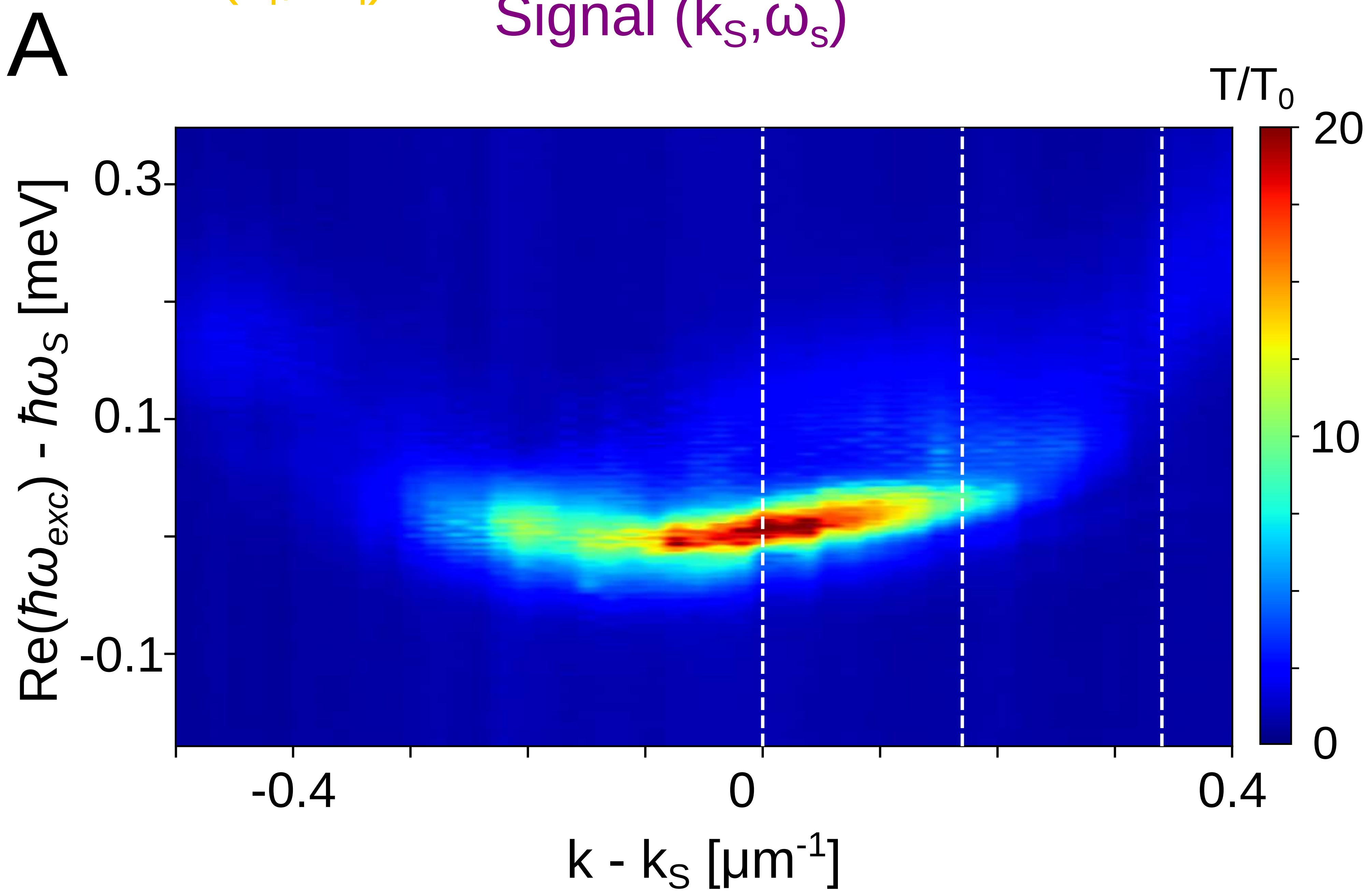


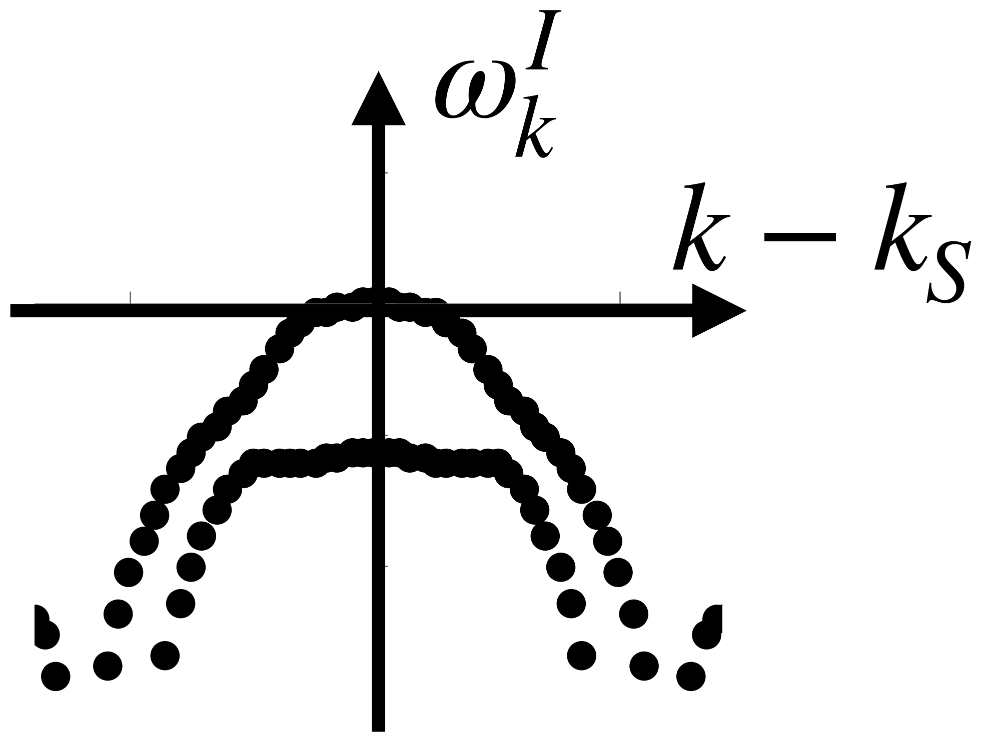
MW and I. Carusotto, PRB 2007

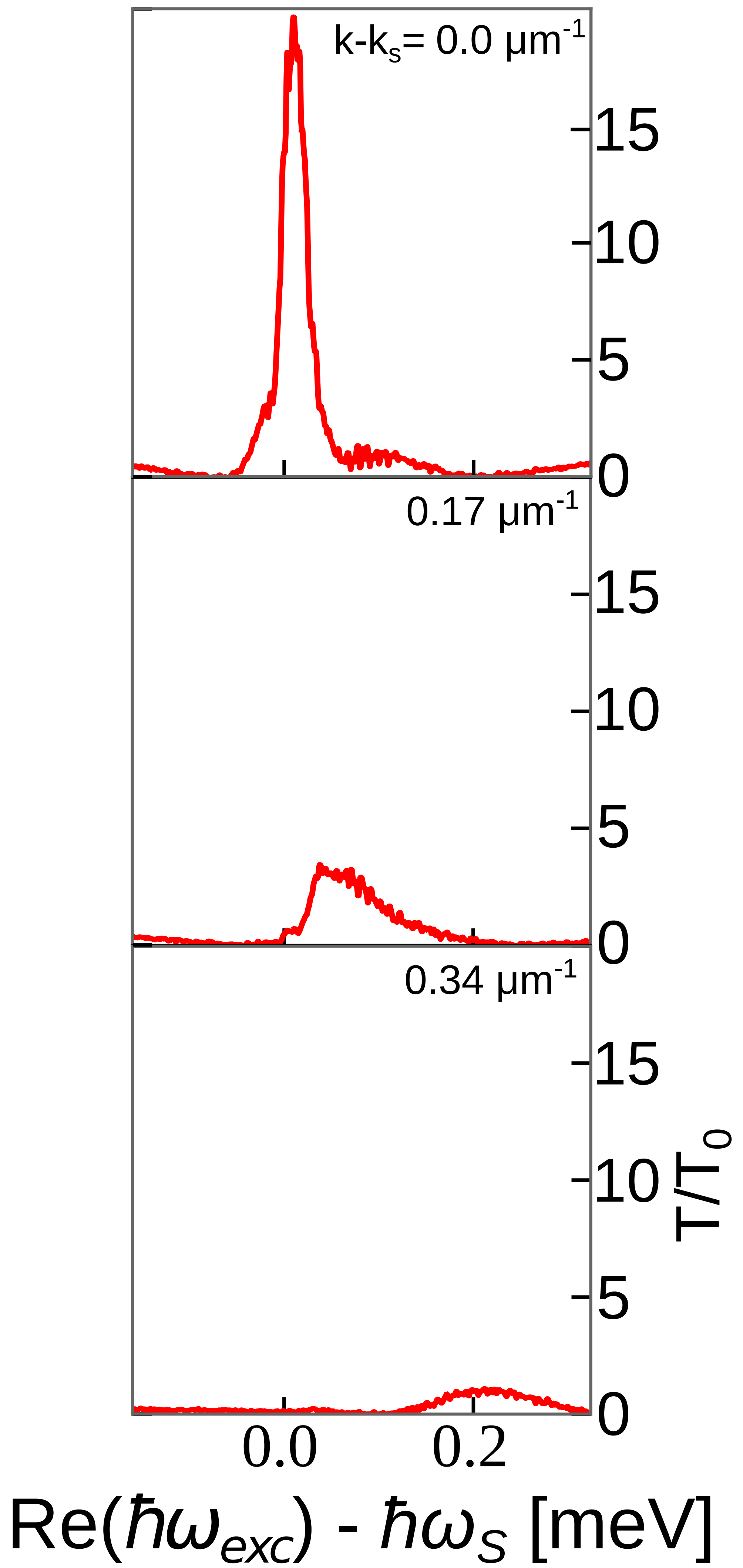
Goldstone measurement



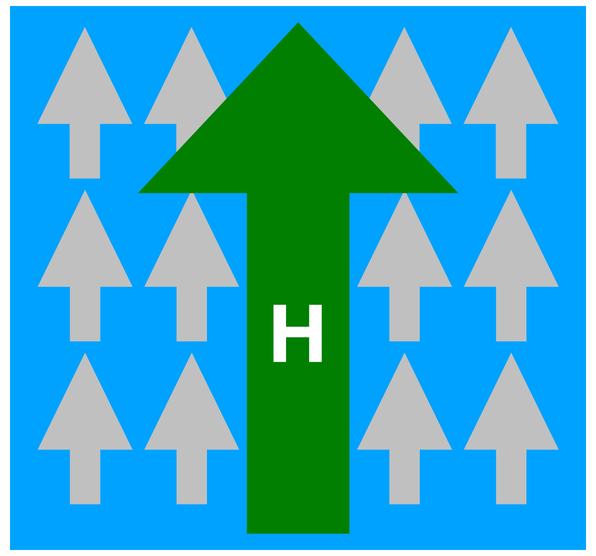
$$T \sim \frac{1}{(\omega - \omega_k^R)^2 + (\omega_k^I)^2}$$

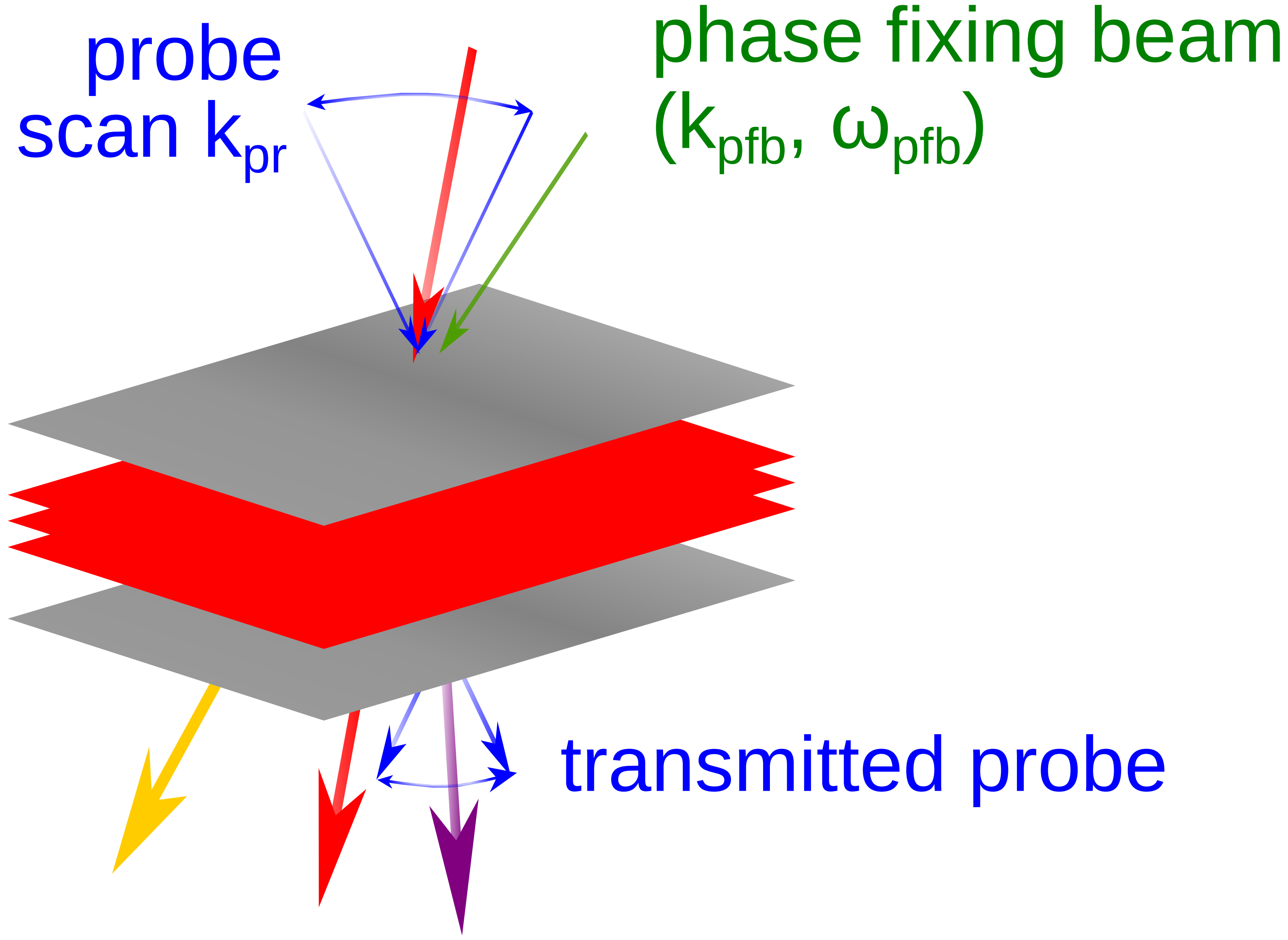


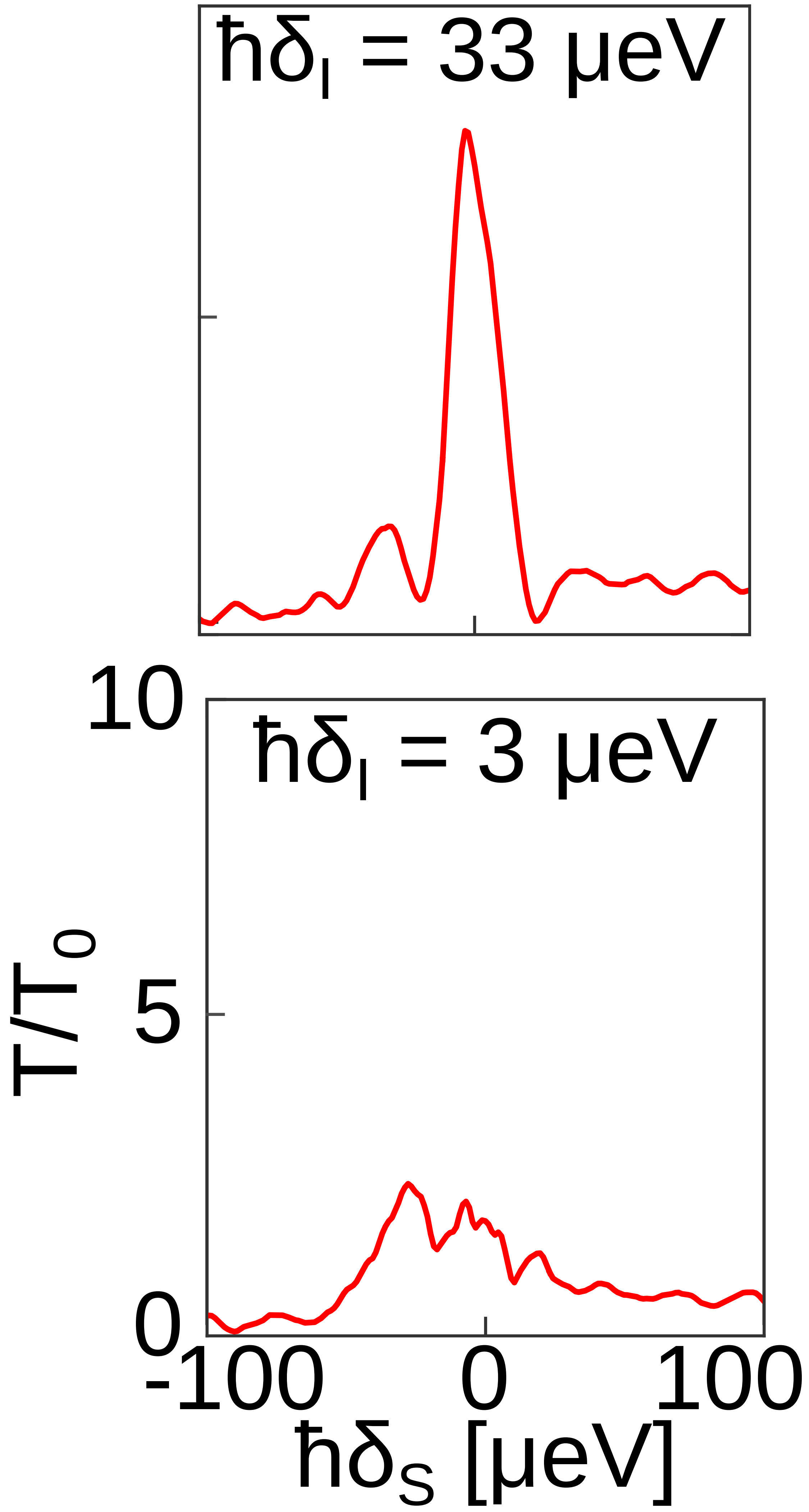




Destroying the Goldstone mode







F. Claude et al. Nature Physics **21**, 924 (2025).

Spatiotemporal coherence in quantum fluids of light

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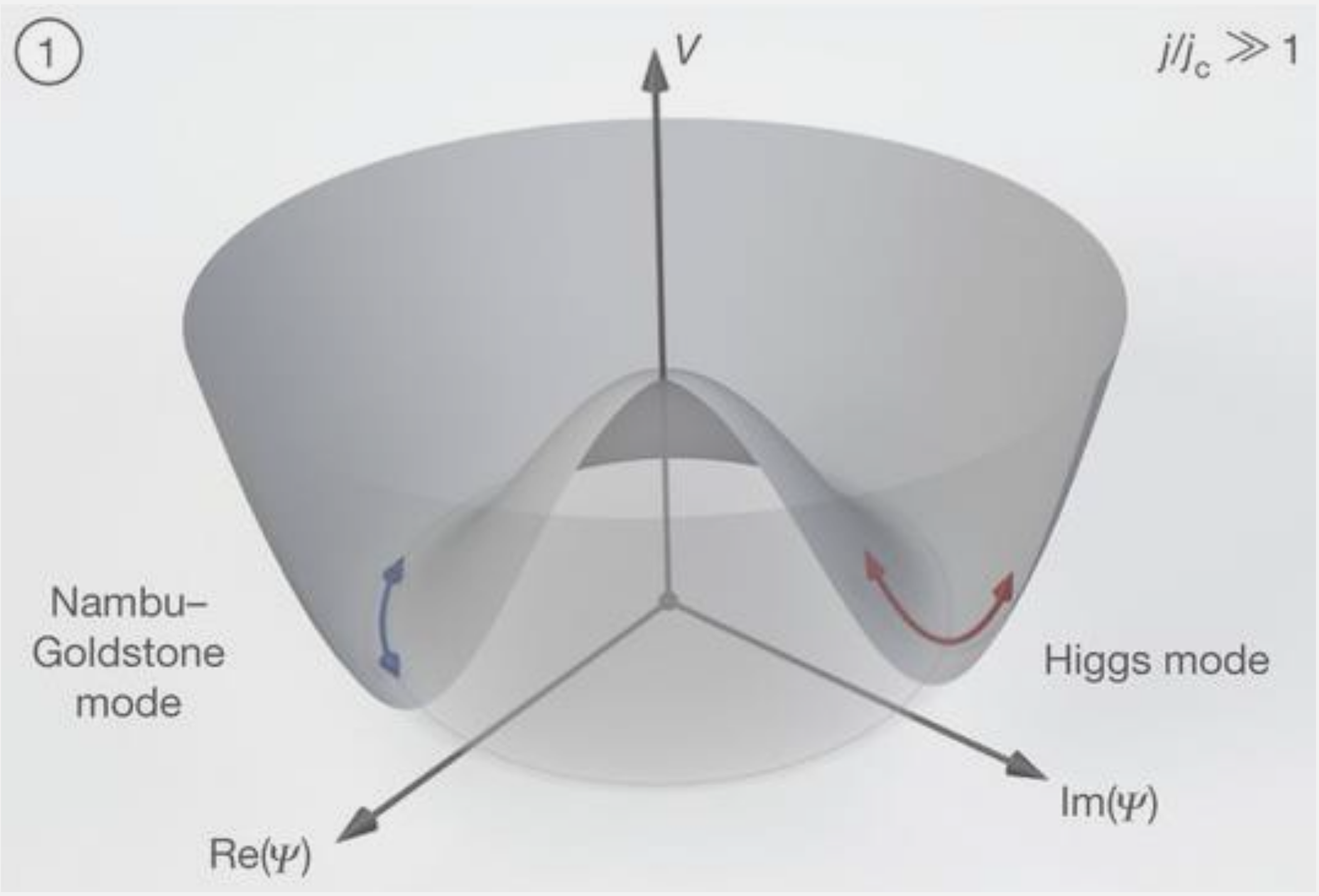
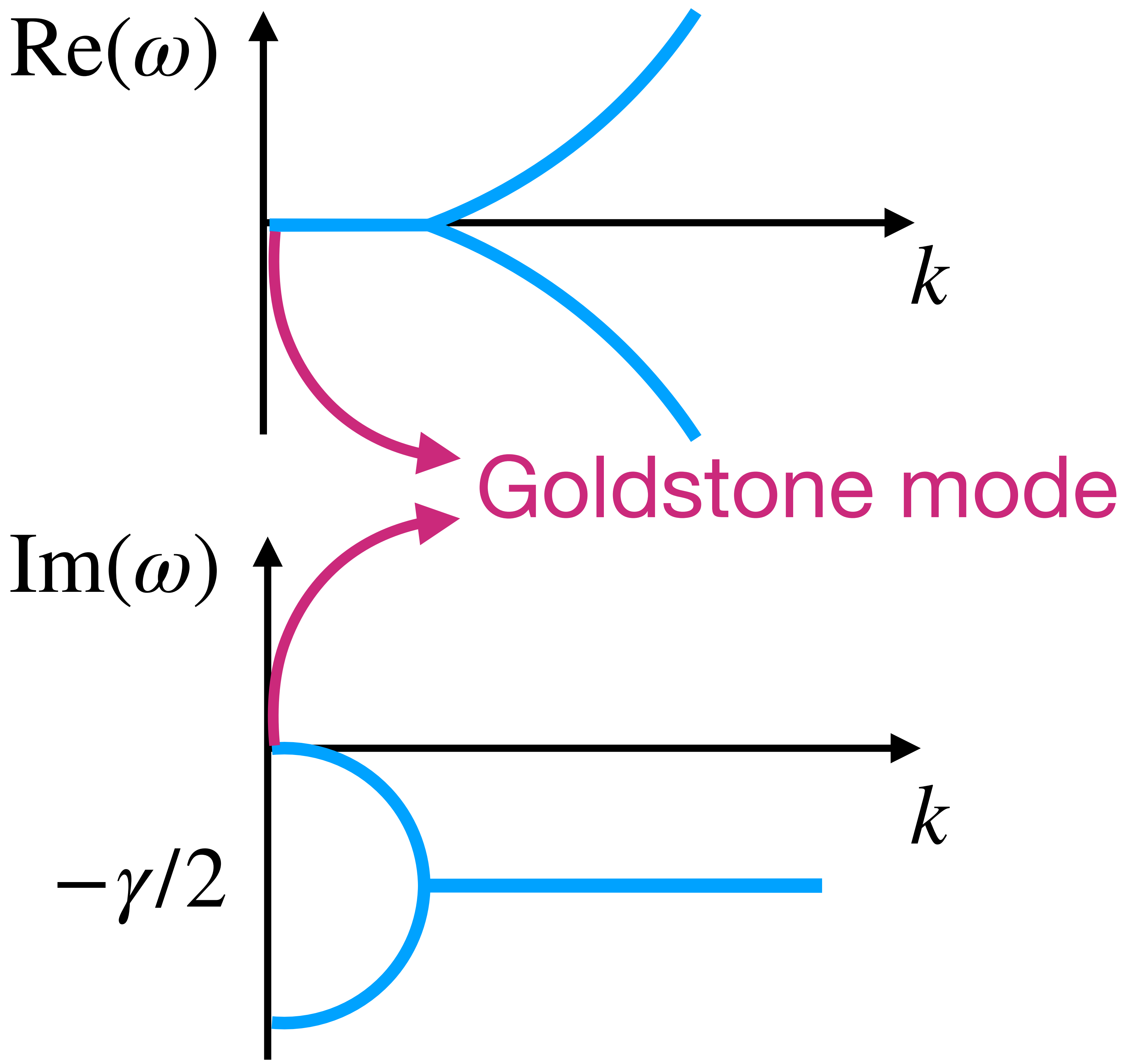
linearized eqs. of mot. in u_k and v_k

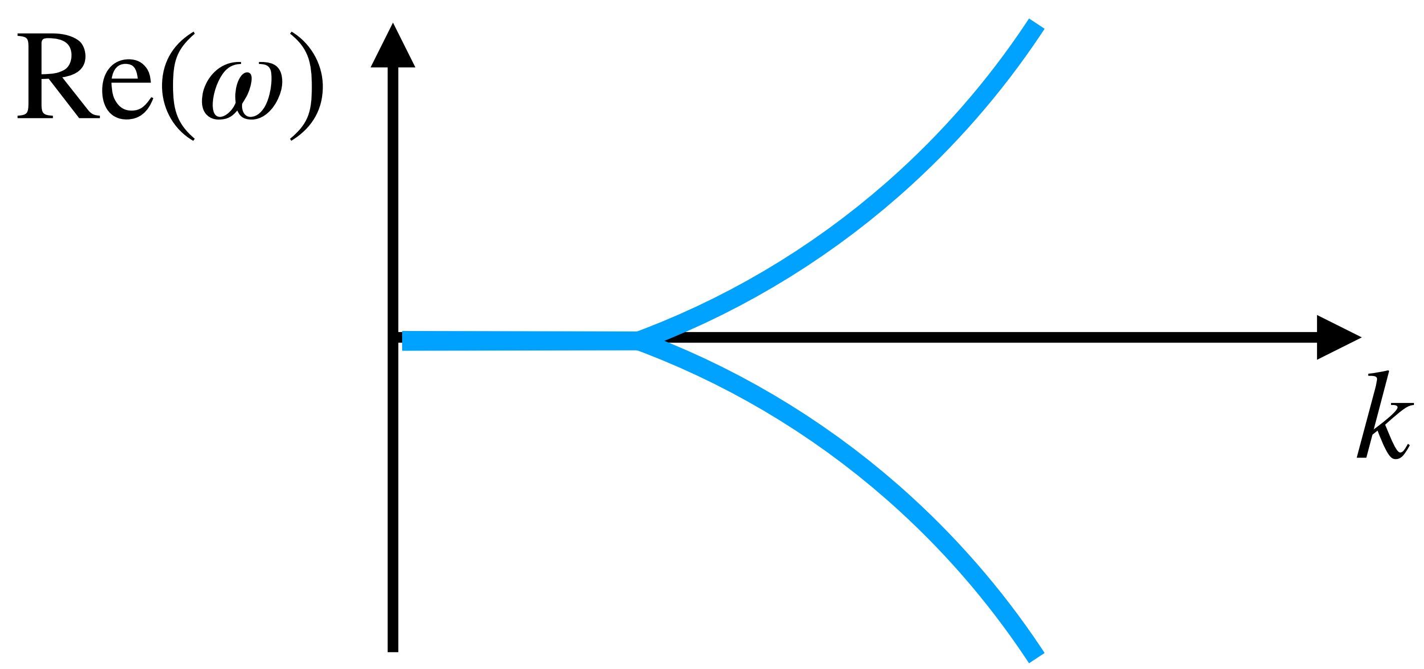


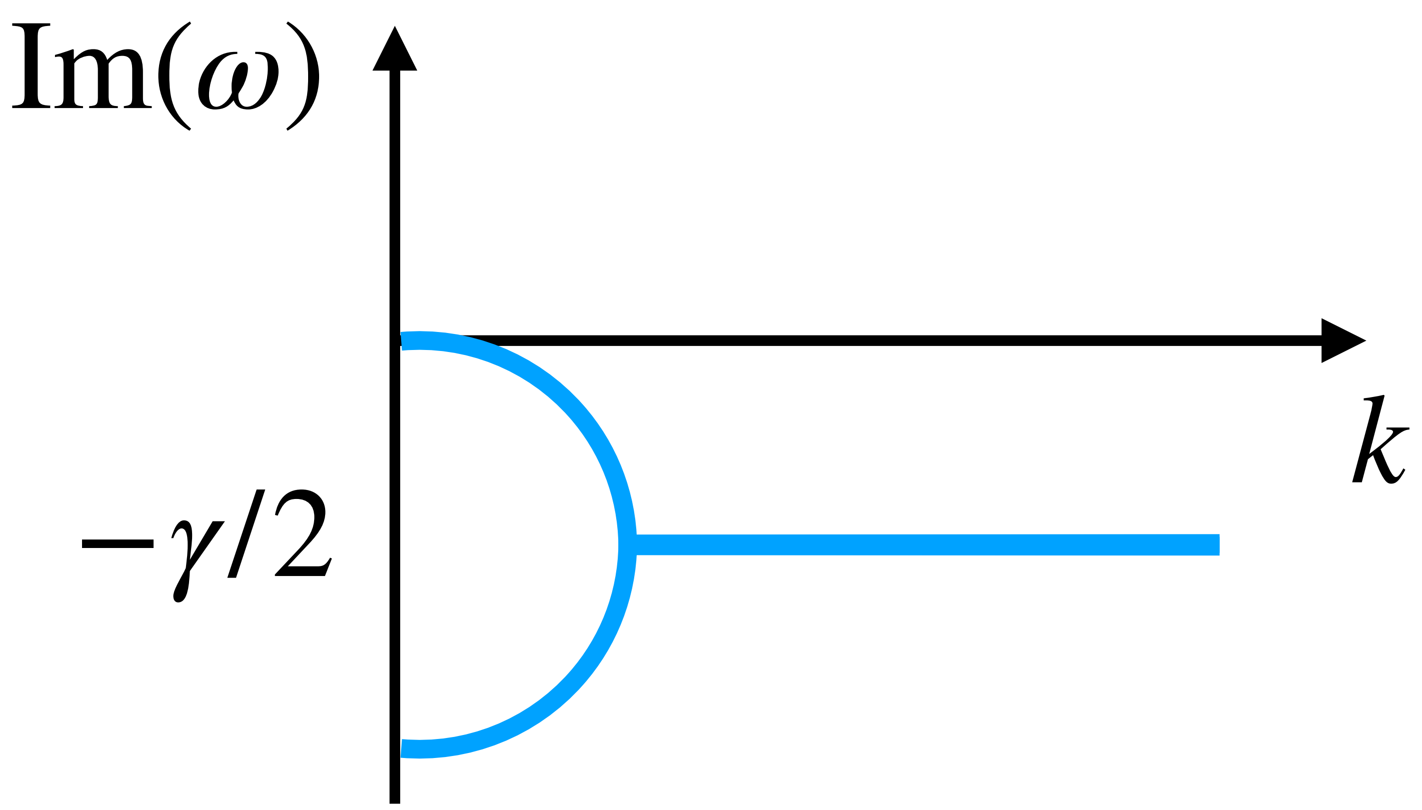
$$i\frac{\partial}{\partial t}\psi(x,t) = -\frac{\nabla^2}{2m}\psi(x,t) + g\,|\psi(x,t)|^2 + g_R n_R(x,t)\,\psi(x,t) + \frac{i}{2}\{R[n_R(x,t)] - \gamma\}\,\psi(x,t) + \sqrt{\frac{R+\gamma}{4\Delta x}}\xi(x,t)$$

Above threshold



Stochastic phase dynamics





$$d\theta(x,t) = a\,\nabla^2\theta(x,t)\;dt + \sqrt{\frac{R}{2n\Delta x}}dW_\theta(x)$$

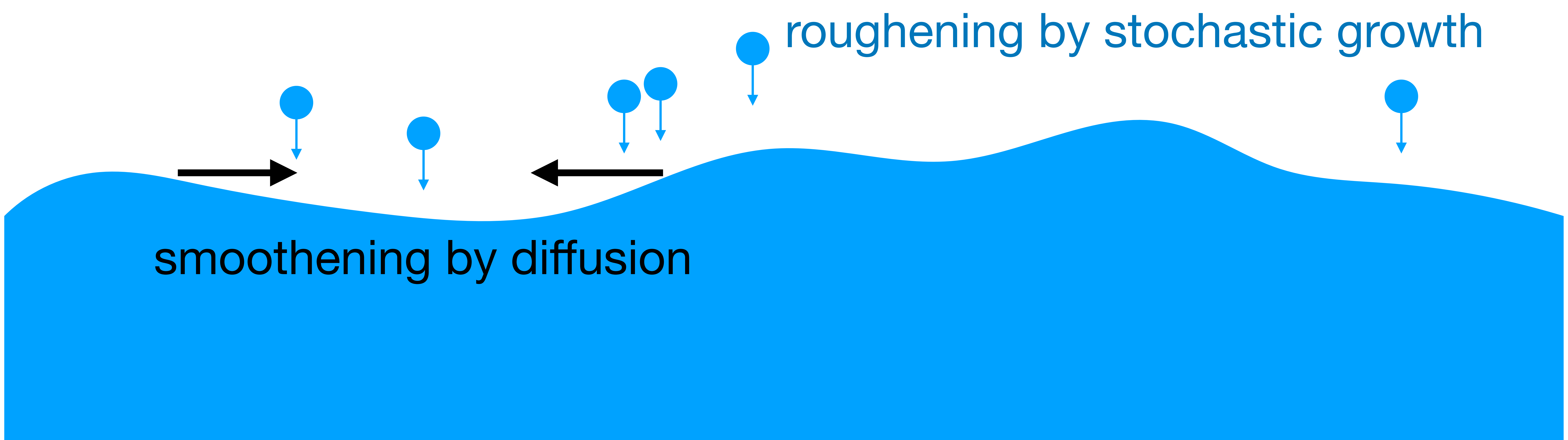
noisy diffusion equation

Stochastic phase dynamics

$$d\theta(x,t) = a\,\nabla^2\theta(x,t)\;dt + \sqrt{\frac{R}{2n\Delta x}}dW_\theta(x)$$

noisy diffusion equation

(compact) Edwards-Wilkinson (EW) model for stochastic interface growth



Phase correlations in EW (Bog.) approx.

The Langevin equation

$$d\theta(x,t) = a\,\nabla^2\theta(x,t)\;dt + \sqrt{\frac{R}{2n\Delta x}}dW_\theta(x)$$

reaches in the steady state the equilibrium distribution of

$$H[\theta] = \int d^D x\;\frac{a}{2}(\nabla\theta)^2$$ at temperature $$T = \sqrt{\frac{R}{4na}}$$

$\Rightarrow$ The same correlation functions as in equilibrium systems

Hohenberg and Halperin, RMP 1977

1D Phase correlator in Bogoliubov (EW) approx.

equal time

$$\langle(\theta(x,t)-\theta(0,t))^2\rangle \sim |x|$$

$$\Rightarrow \quad \langle\psi^*(x,t)\psi(0,t)\rangle \approx n e^{-\frac{1}{2}\langle[\theta(x,t)-\theta(0,t)]^2\rangle} \sim e^{-|x|/\ell_c}$$

equal space

$$\langle(\theta(x,t)-\theta(x,0))^2\rangle \sim \sqrt{|t|}$$

$$\Rightarrow \quad \langle\psi^*(x,t)\psi(x,0)\rangle \approx n e^{-\frac{1}{2}\langle[\theta(x,t)-\theta(x,0)]^2\rangle} \sim e^{-\sqrt{|t|/\tau_c}}$$

Spatiotemporal scaling

$$C_\theta(x,t) = \langle [\theta(x,t) - \theta(x+\Delta x, t+\Delta t)]^2 \rangle \sim \Delta t^{2\beta} F\left(y_0 \frac{\Delta x}{\Delta t^{1/z}} \right)$$

α: roughening exponent
z: dynamical exponent
$\beta = \alpha/z$: growth exponent

EW: $\alpha = \dfrac{2-d}{2}$

$\beta = \dfrac{2-d}{4}$

$z = 2$

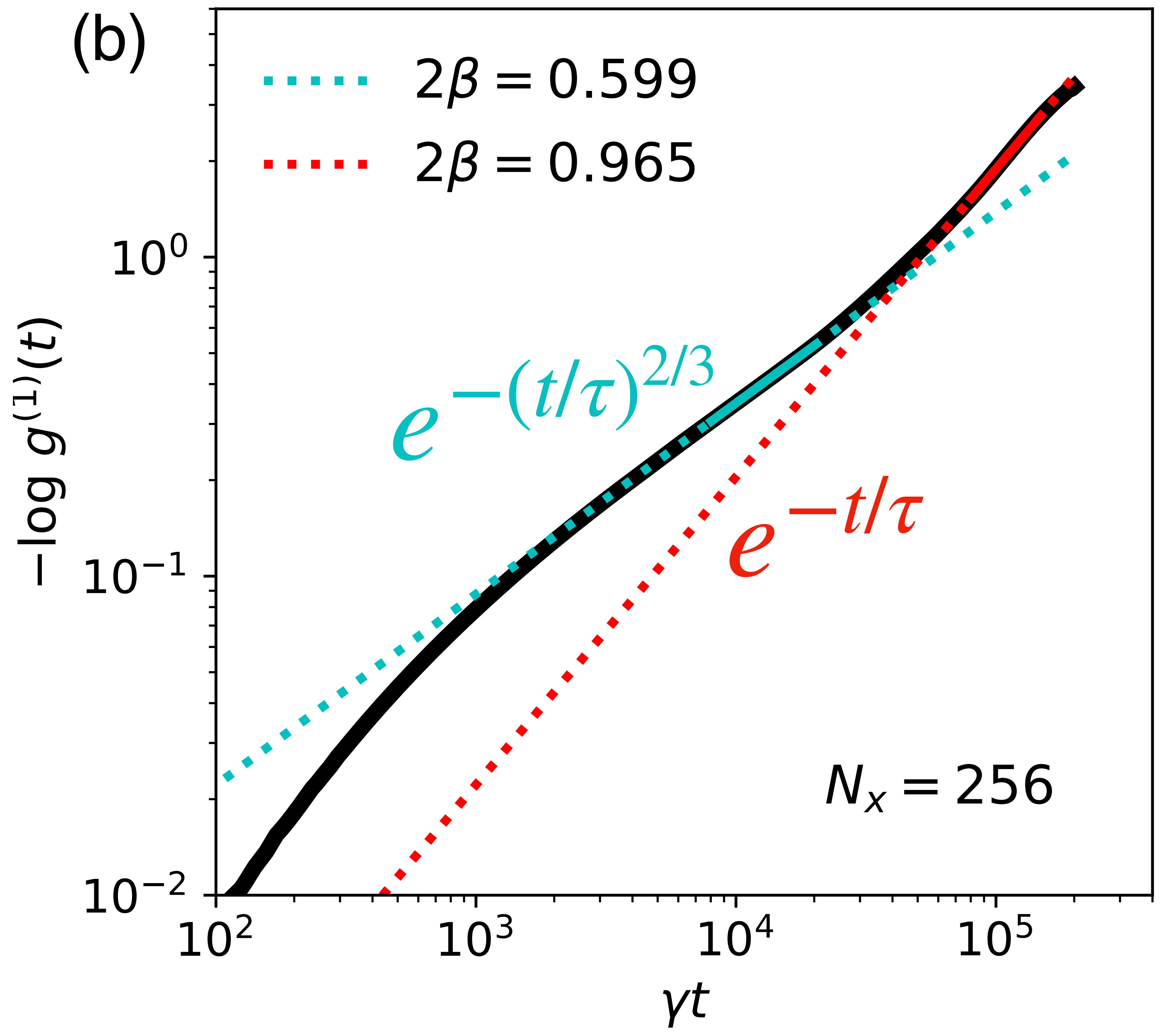
with $F(y \to 0) \to c^{te}$

$F(y \to \infty) \sim |y|^{2\alpha}$

$C_\theta \sim \Delta x^{2\alpha}$

$C_\theta \sim \Delta t^{2\beta}$

Long times: Schawlow-Townes diffusion



Finite system: mode quantization → crossover to 0D system at $t \gg \Delta\epsilon$

I. Amelio and I. Carusotto, PRX **10**, 041060 (2020)

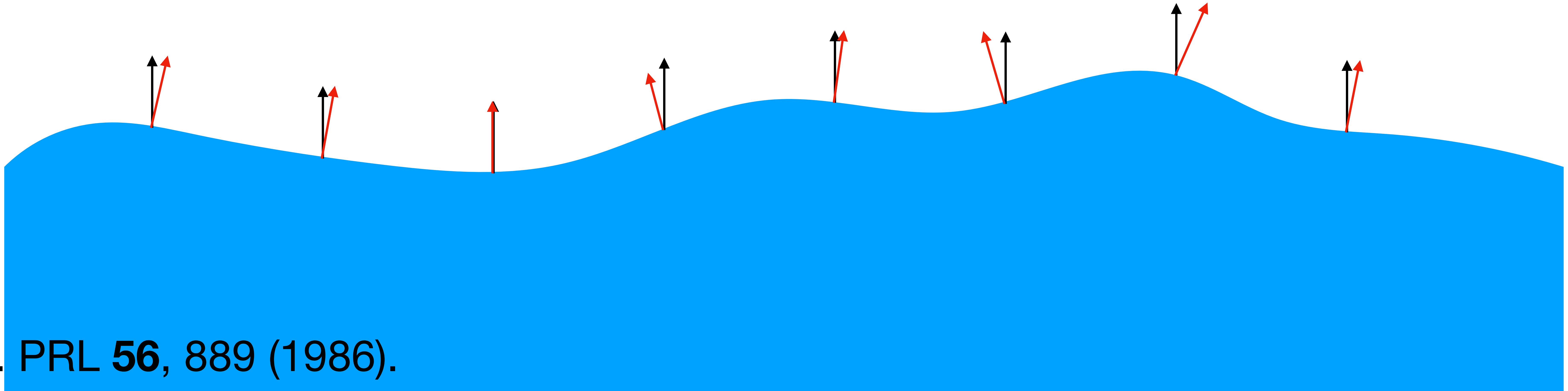
Kardar-Parisi-Zhang

EW vertical interface growth

KPZ growth normal to surface

$$d\theta = [a\nabla^2\theta + \lambda(\nabla\theta)^2]\, dt + \sqrt{\frac{R}{2n\Delta x}} dW_\theta$$

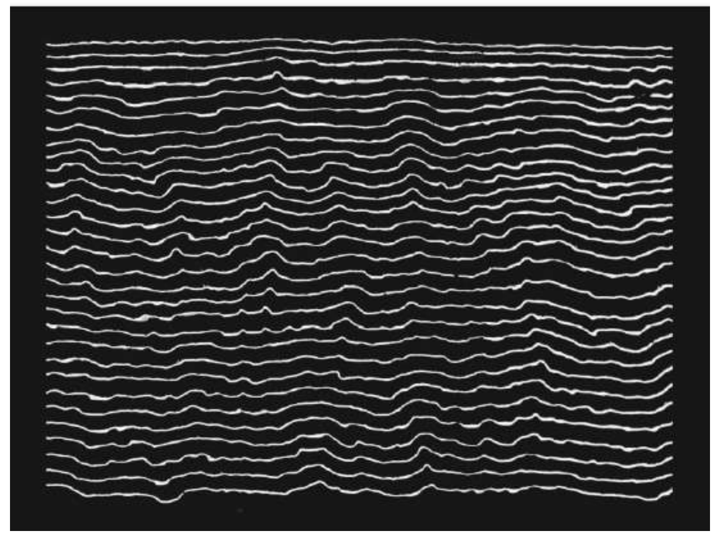
nonequilibrium, increases fluctuations



M. Kardar et al. PRL **56**, 889 (1986).

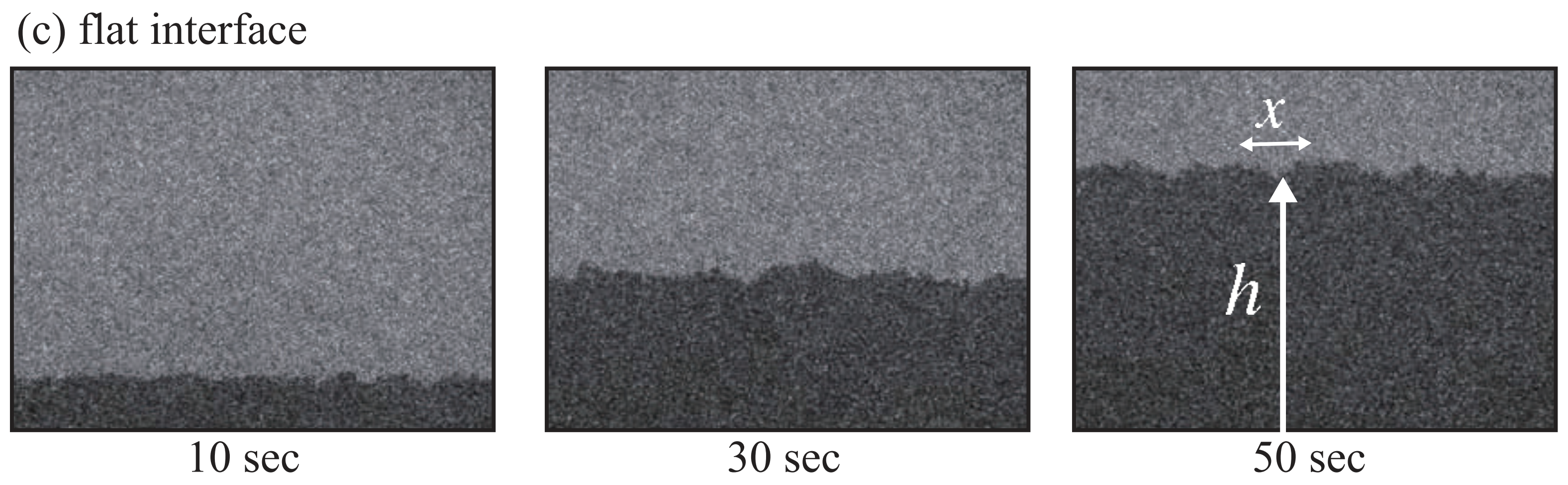
KPZ examples

Paper combustion



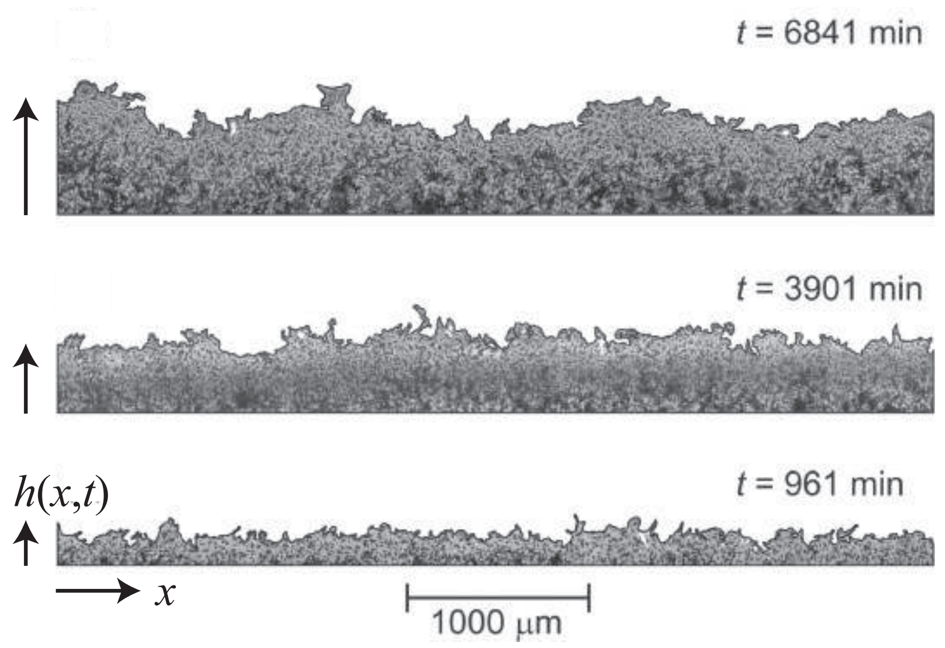
M. Myllys et al. PRE 64, 036101 (2001)

Liquid crystal interface



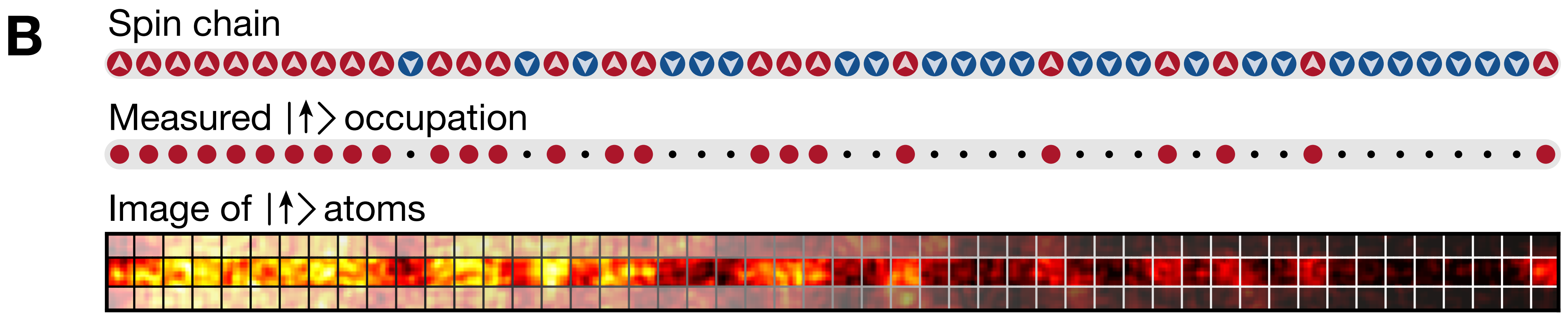
K. A. Takeuchi, M. Sano, J. Stat. Phys. 147 (2012) 853

Cancer cells



M.A.C.Huergo et al., Phys. Rev. E 85 (2012) 011918.

Ultracold atoms



D. Wei et al. Science Vol 376, pp. 716-720 (2022).

Phase nonlinearity in noneq. quantum fluids

Josephson: nonlinear contribution to phase dynamics from kinetic energy:

$$d\theta(x,t) = [a\nabla^2\theta(x,t) - \frac{1}{2m}(\nabla\theta)^2]\,dt + \sqrt{\frac{R}{2n\Delta x}}dW_\theta(x)$$

Spatiotemporal coherence at long distances/times

$$\langle\psi^*(x,t)\psi(x',t')\rangle \approx n e^{-\frac{1}{2}\langle[\theta(x,t)-\theta(x',t')]^2\rangle}$$

has KPZ scaling properties

1D KPZ

$$\log\left[g^{(1)}(\Delta x, \Delta t)\right] \sim \langle[\theta(x,t) - \theta(x+\Delta x, t+\Delta t)]^2\rangle \sim \Delta t^{2\beta} F_{KPZ}\left(y_0 \frac{\Delta x}{\Delta t^{1/z}}\right)$$

computed exactly by Prähofer & Spohn

$$\log\left[g^{(1)}(\Delta x, 0)\right] \sim \Delta x^{2\alpha} = \Delta x$$

$$\log\left[g^{(1)}(0, \Delta t)\right] \sim \Delta t^{2\beta} = \Delta t^{2/3}$$

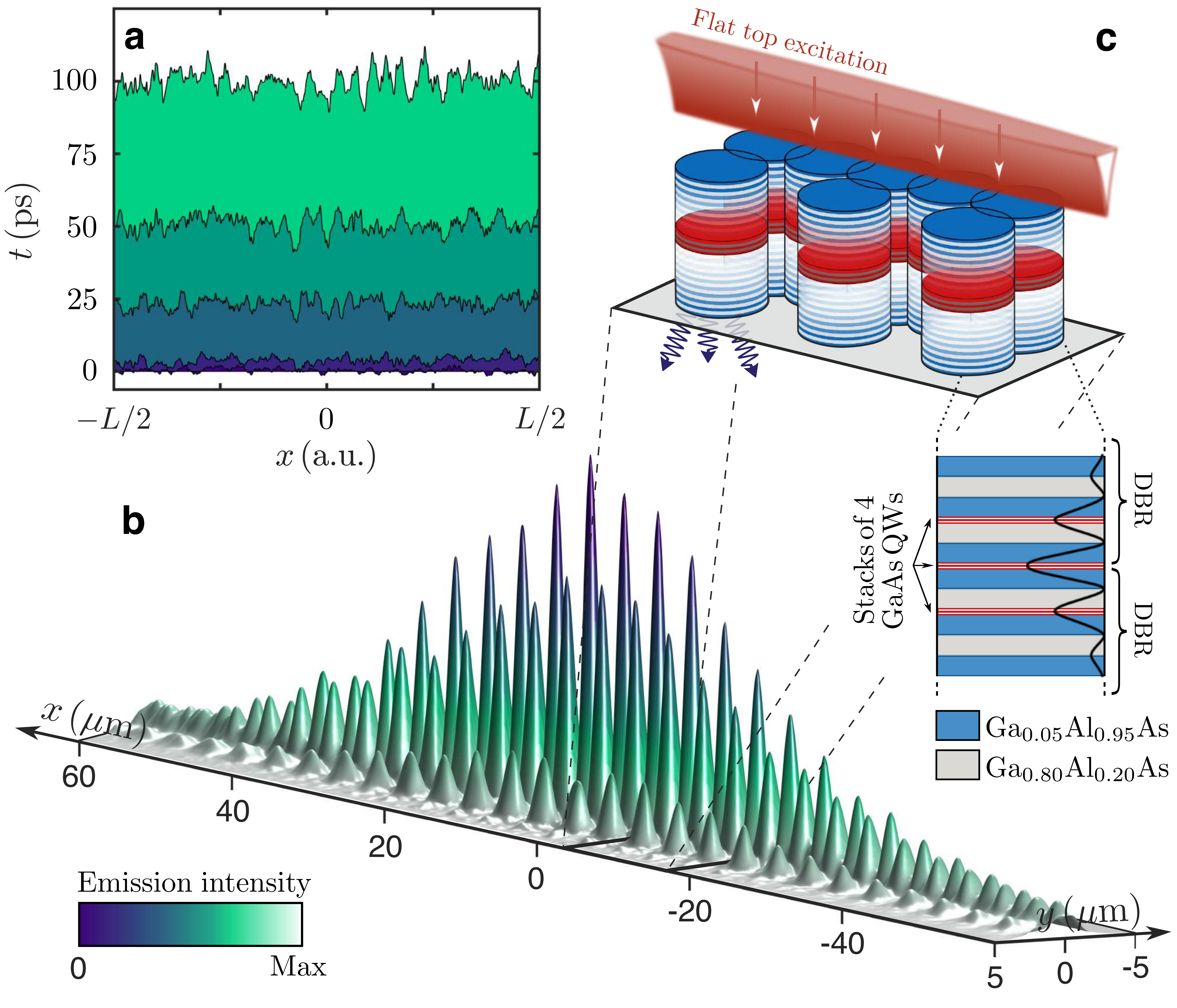
KPZ: $\alpha = 1/2$
$\beta = 1/3$
$z = 3/2$

EW: $\alpha = 1/2$
$\beta = 1/4$
$z = 2$

$\alpha + z = 2$ from Galilean invariance

M. Prähofer and H. Spohn, J. Stat. Phys. **154**, 1191–1227 (2014).

experiment KPZ 1D polaritons



Q. Fontaine et al.

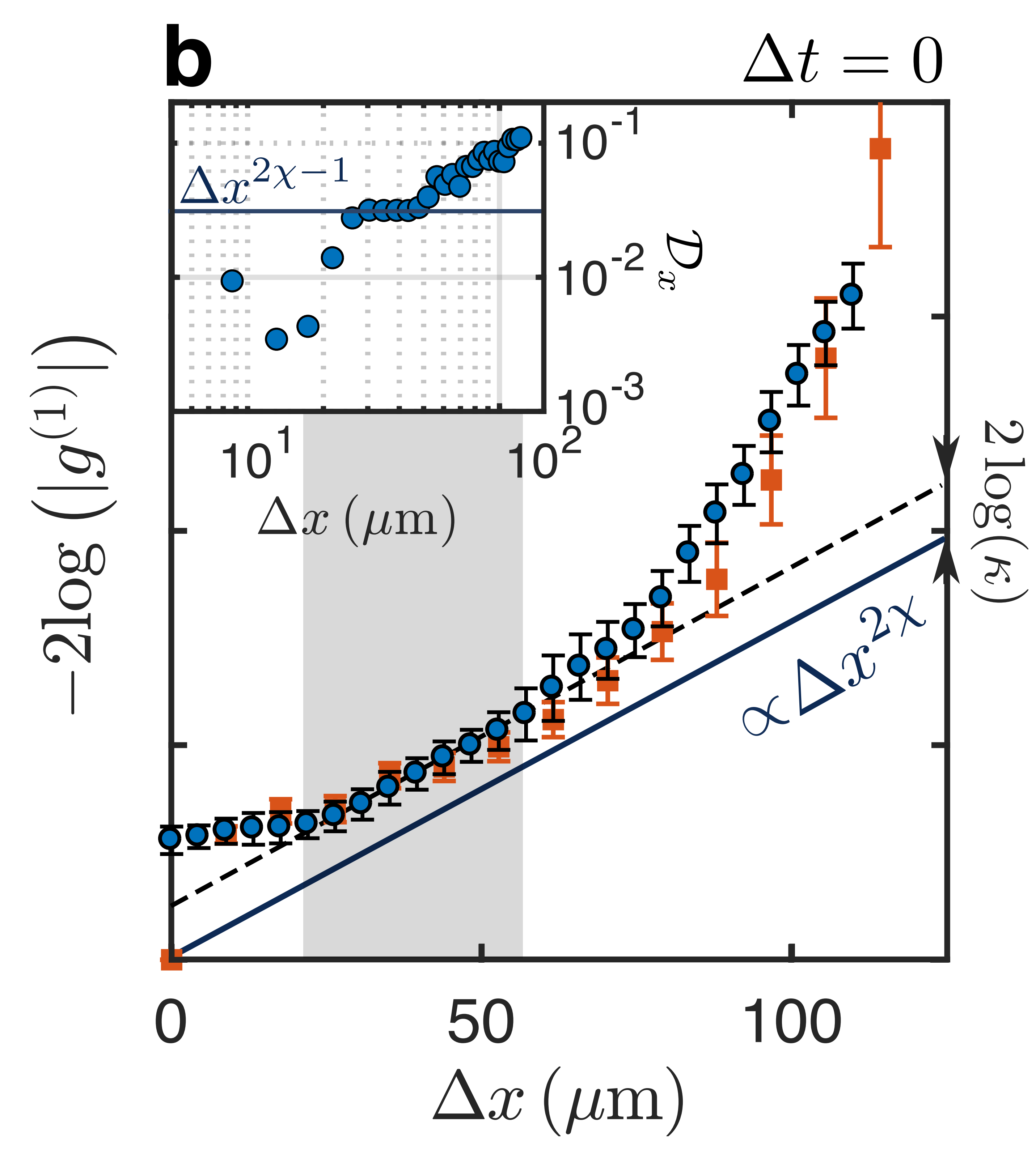
Nature **608**, 687 (2022)

KPZ scaling

$$g^{(1)}(\Delta x, \Delta t) \approx e^{-\frac{1}{2}\langle[\theta(x+\Delta x, t+\Delta t)-\theta(x,t)]^2\rangle}$$

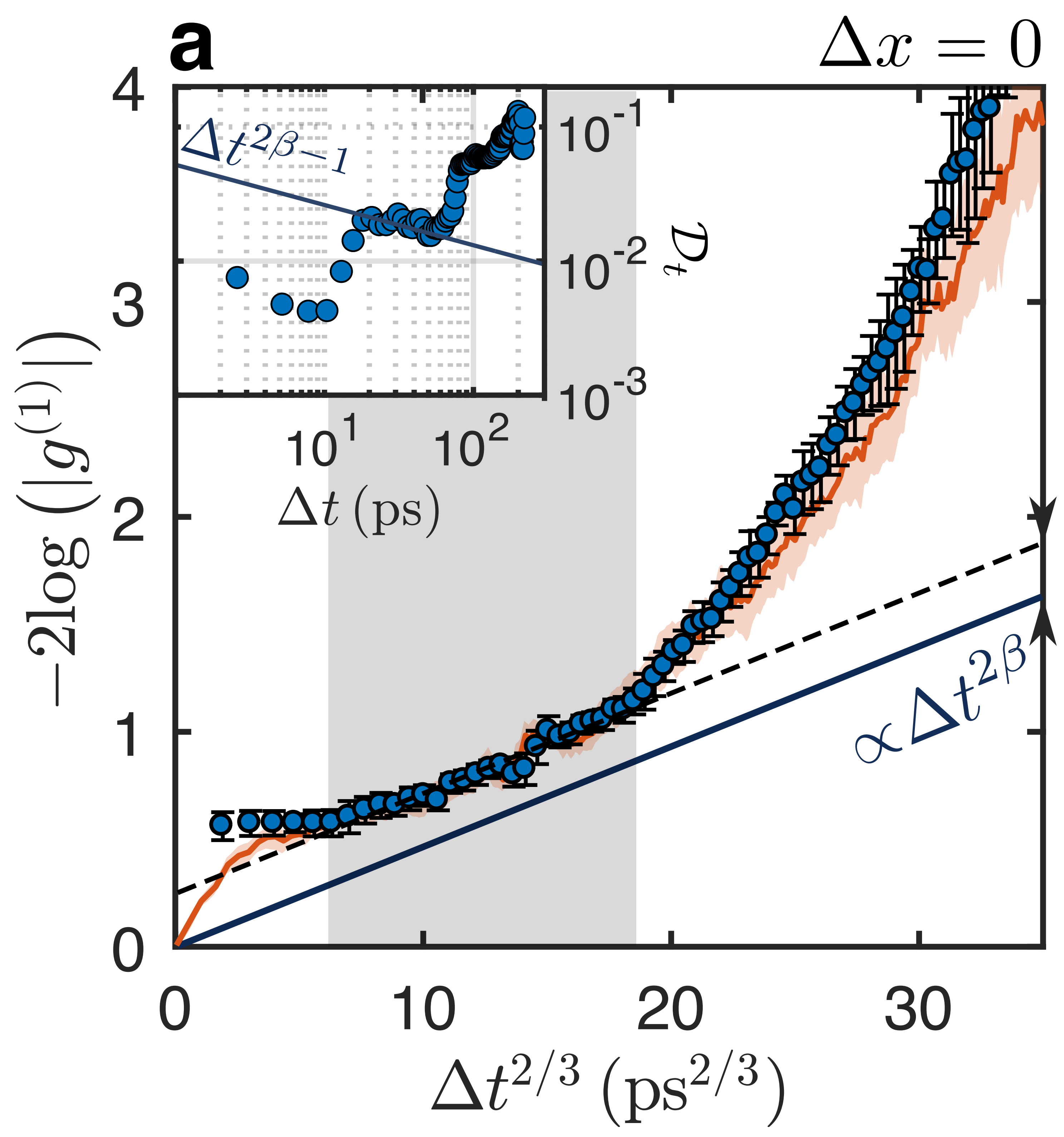
Spatial scaling (same as Bogoliubov)

$$g^{(1)}(\Delta x, \Delta t = 0) \sim e^{-|\Delta x|/\ell}$$



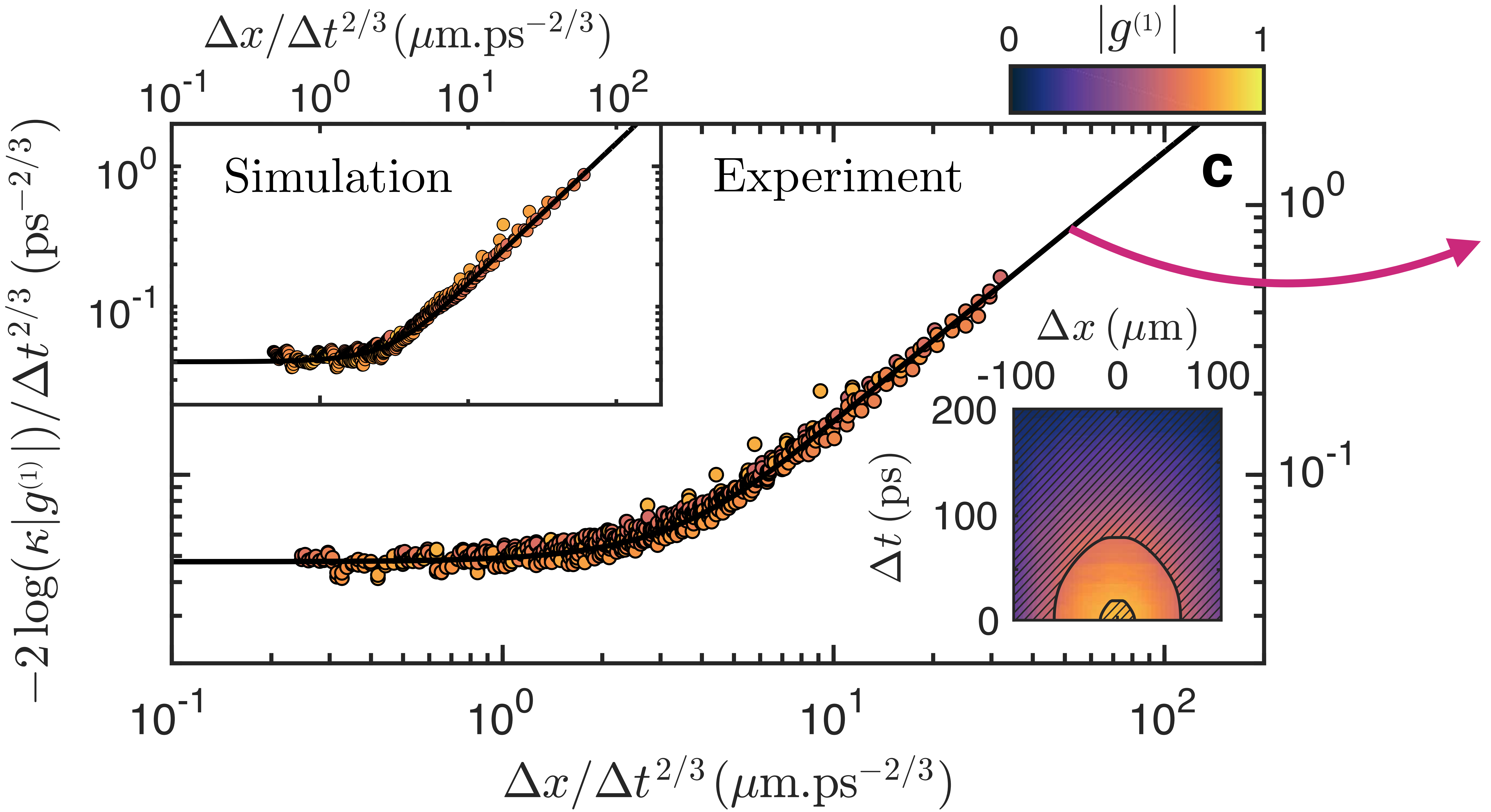
Temporal scaling

$$g^{(1)}(\Delta x = 0, \Delta t) \sim e^{-|\Delta t/\tau|^{2/3}}$$



KPZ scaling

$$-2\log\left[g^{(1)}(\Delta x, \Delta t)\right] / |\Delta t|^{2/3} \sim F(\Delta x/\Delta t^{2/3})$$



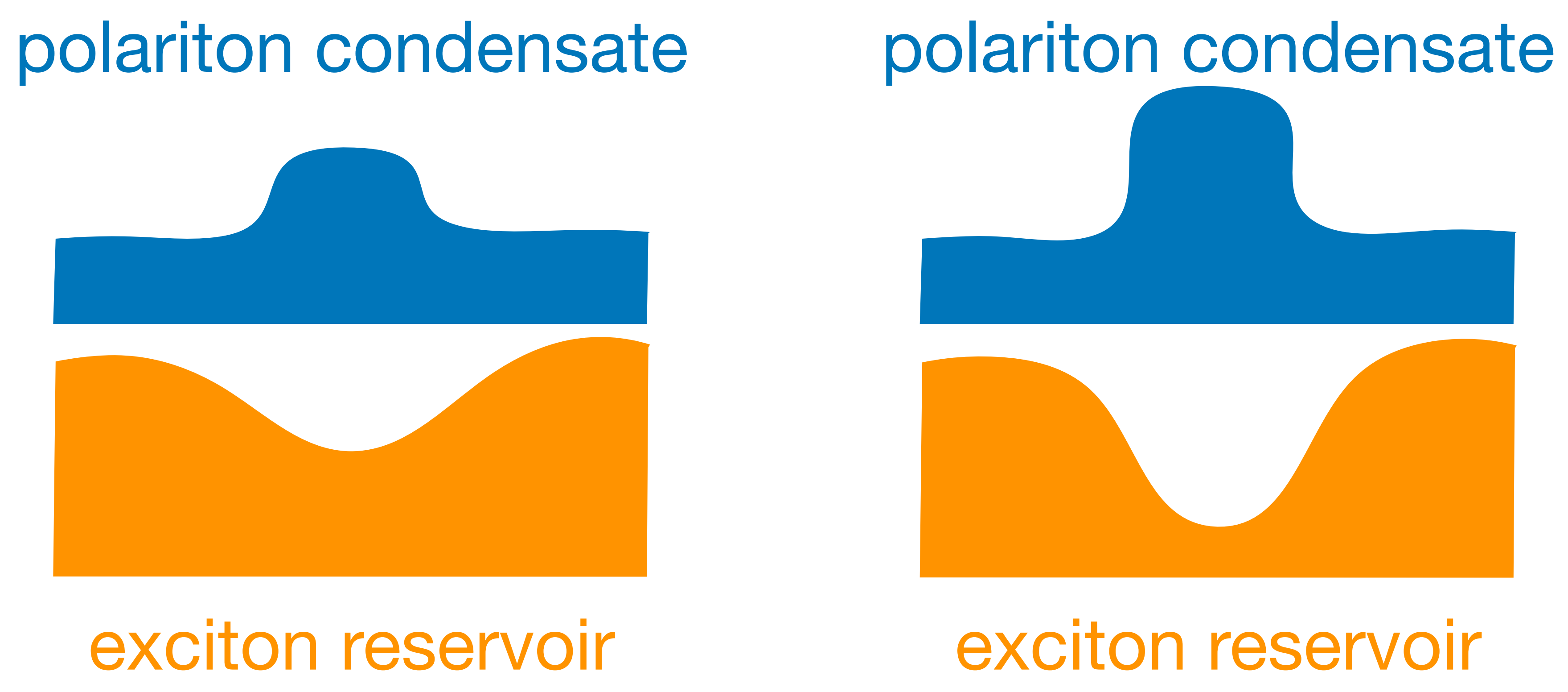
Q. Fontaine et al.

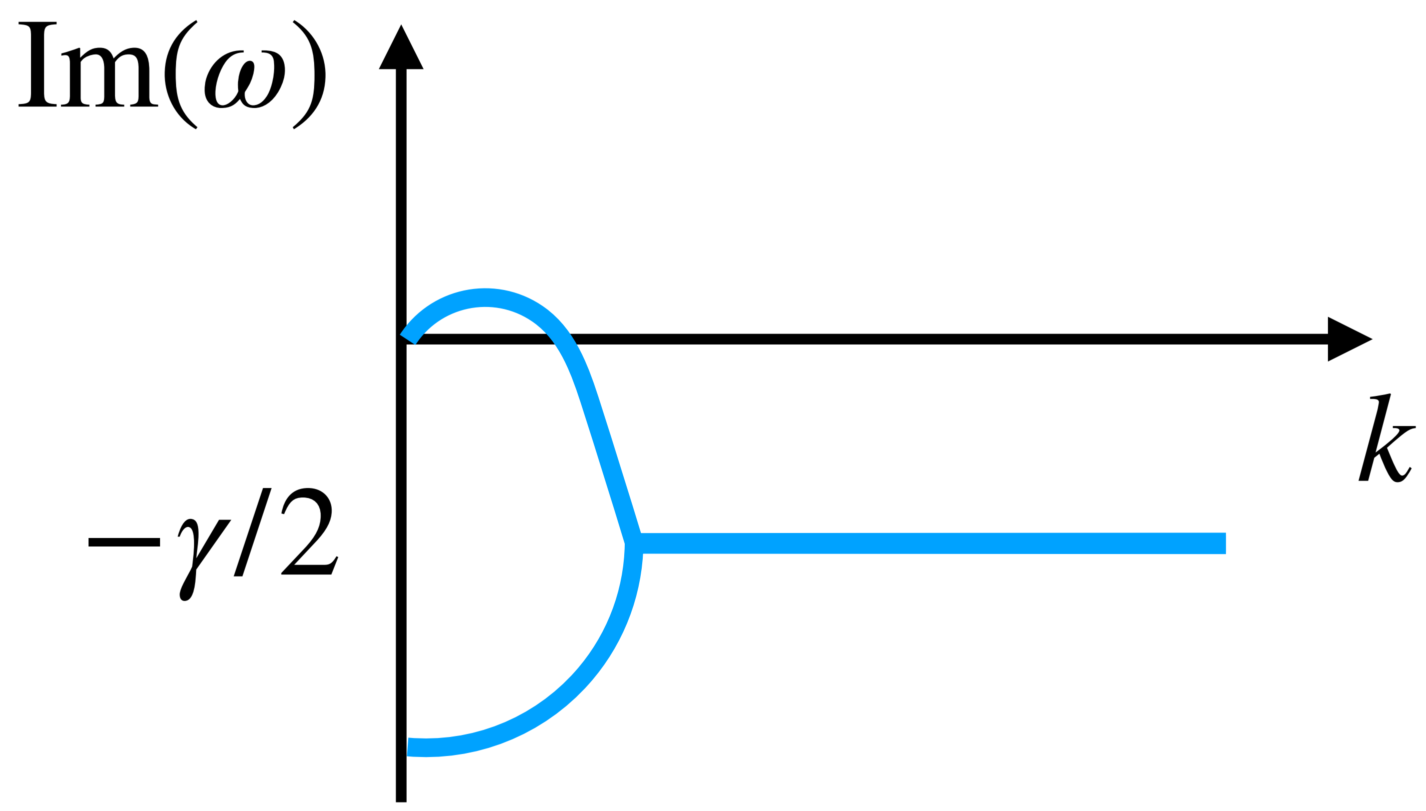
Nature **608**, 687 (2022)

Some more details…

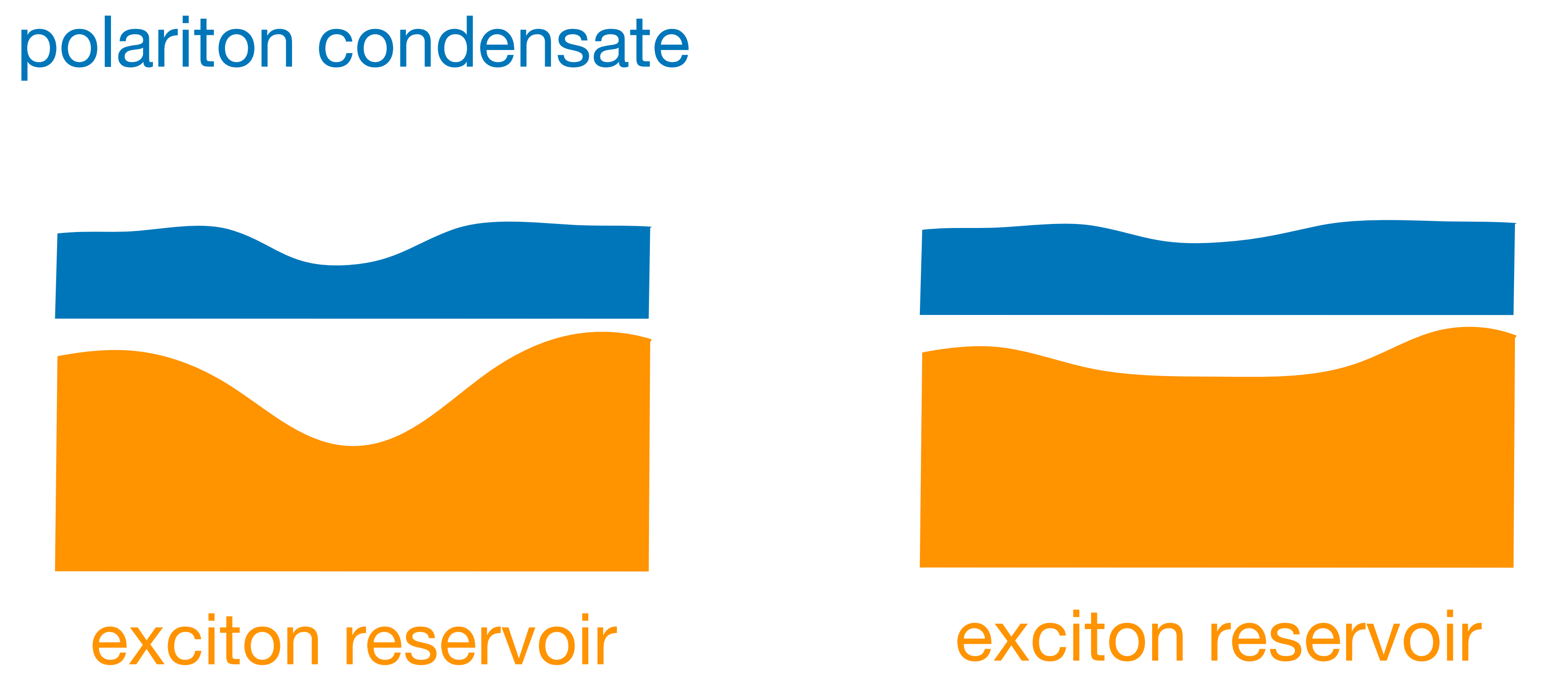
Modulational instability

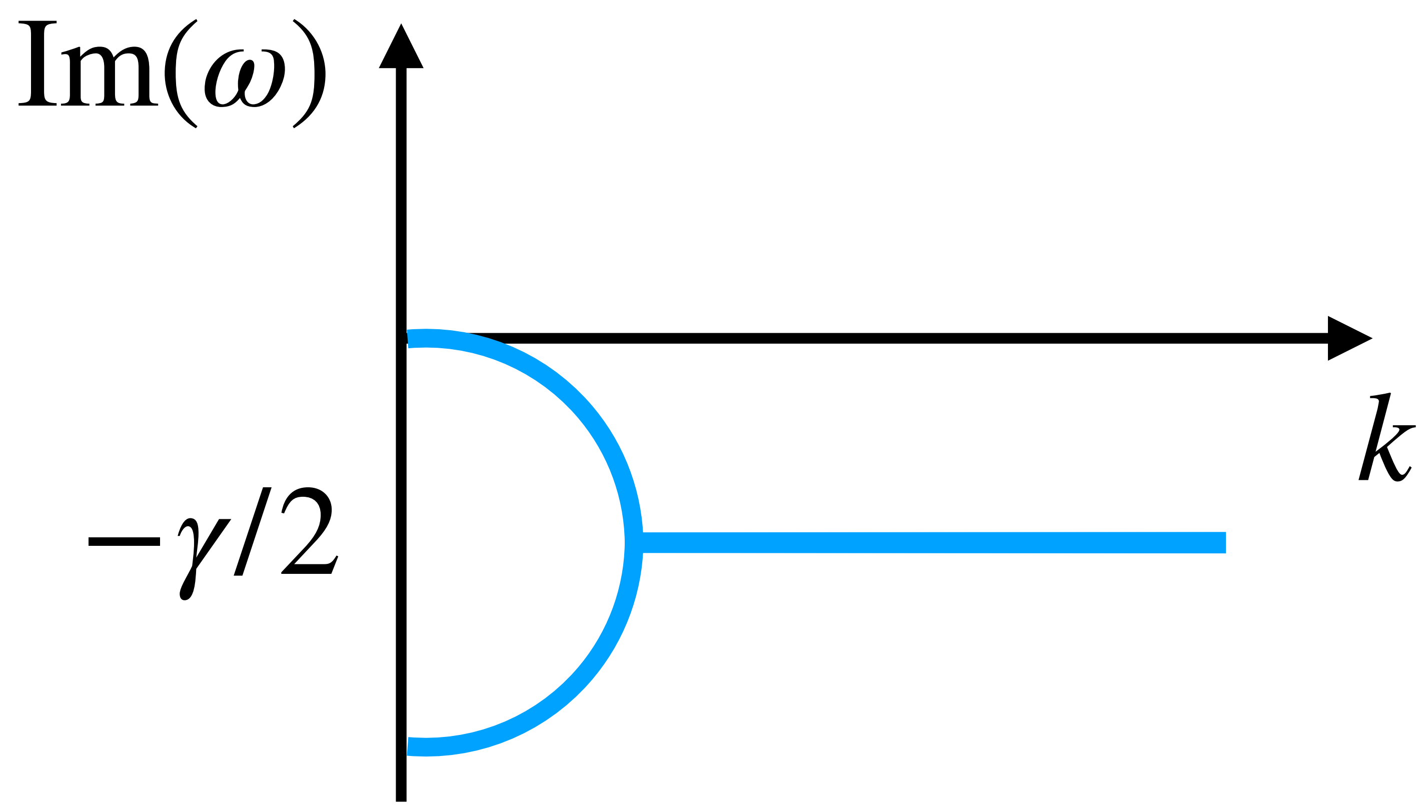
Instability mechanism from interactions with reservoir





Avoided with negative mass states

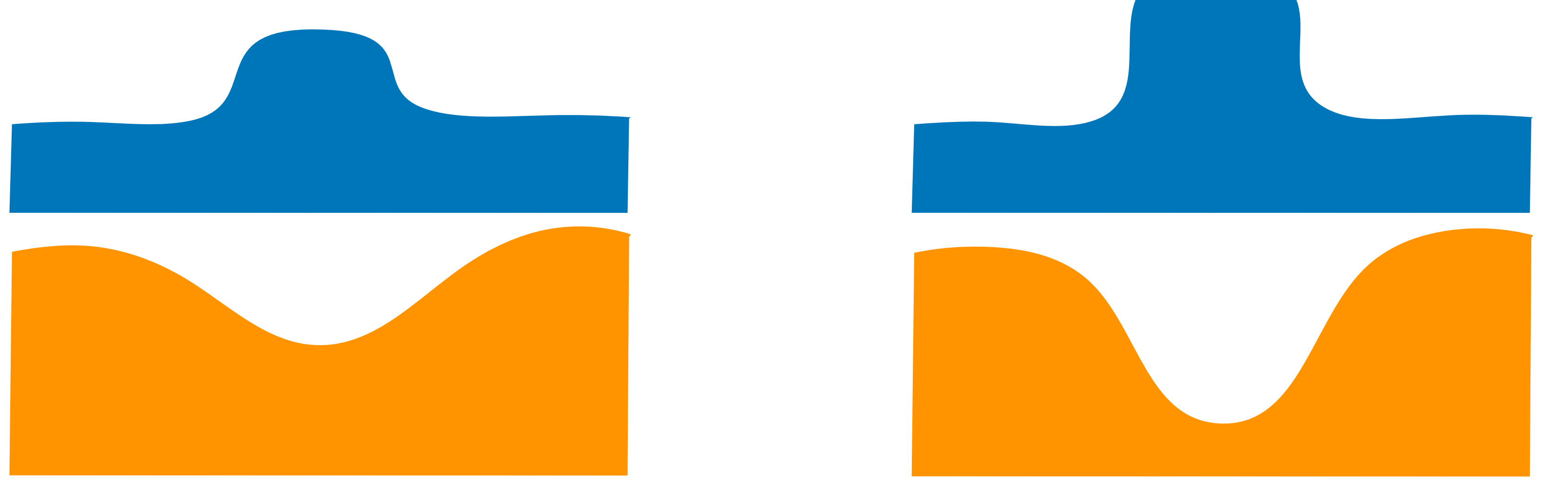




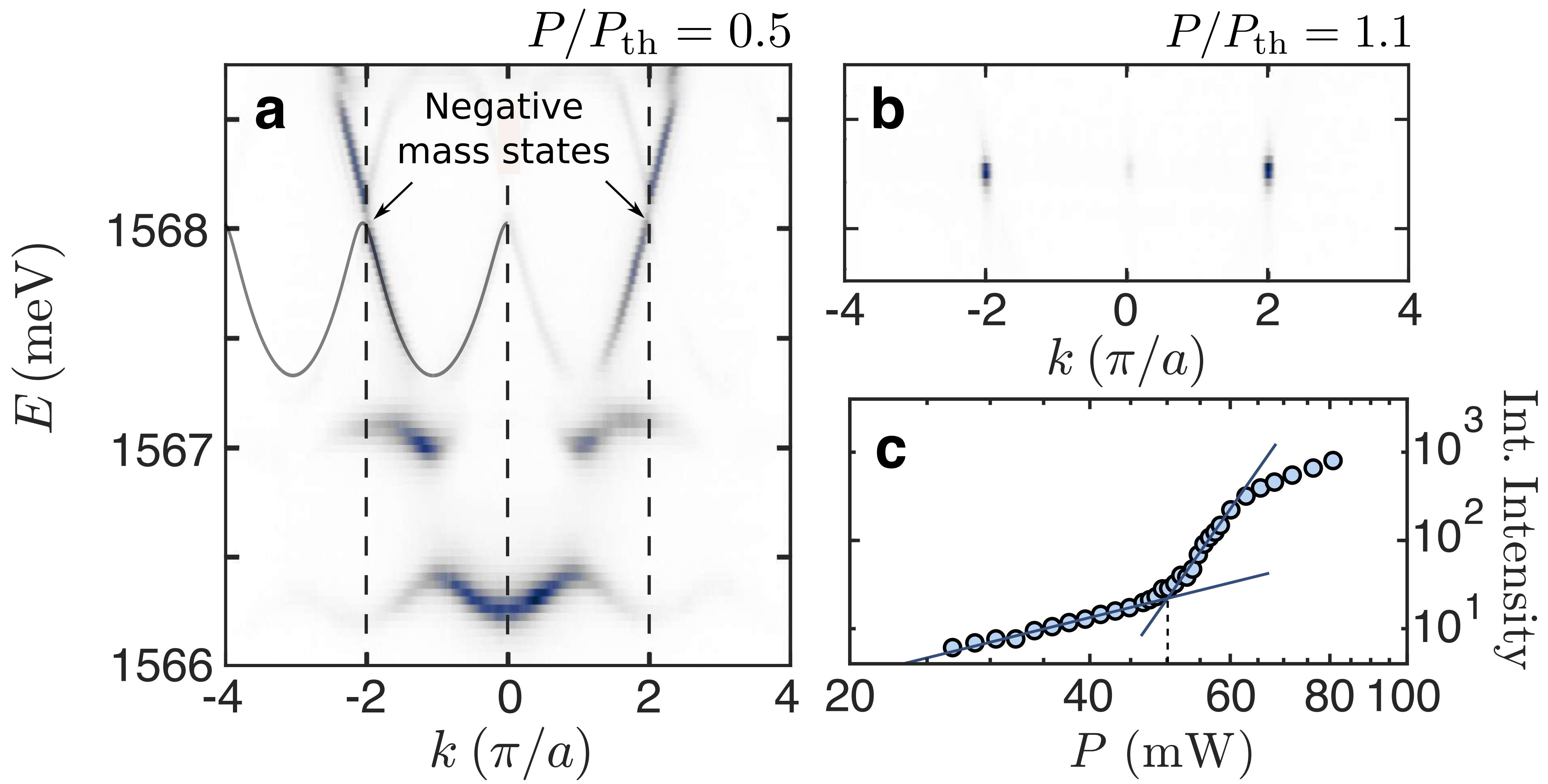
F. Baboux et al. Optica 5, 1163 (2018)

Modulational instability

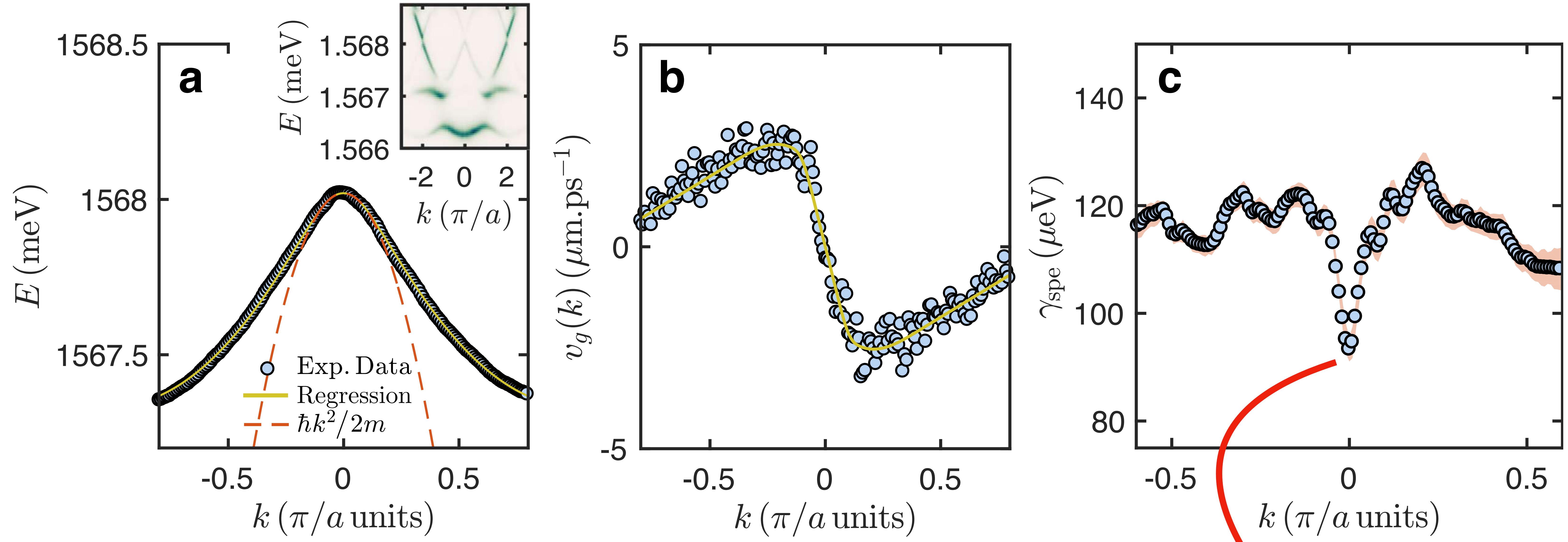
Instability mechanism from interactions with reservoir



Avoided with negative mass states



k-dependent linewidth



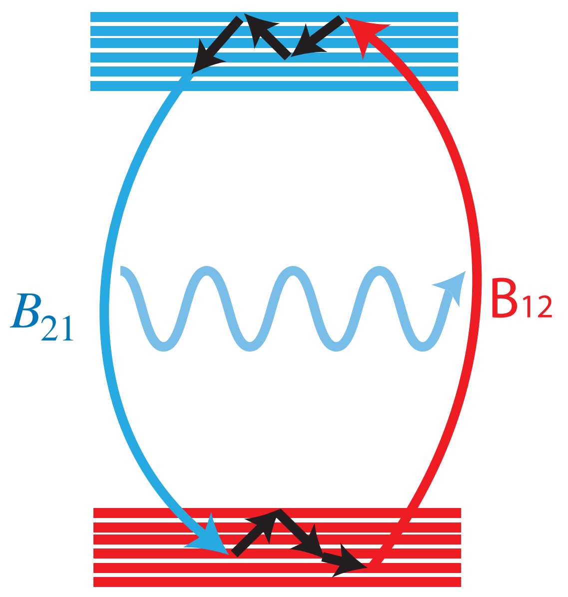
Longer life time for condensate mode

$$i\frac{\partial}{\partial t}\psi(x,t) = -\frac{\nabla^2}{2m}\psi(x,t) + i\gamma_2\nabla^2\psi + \frac{i}{2}\{R[n_R(x,t)] - \gamma\}\psi(x,t) + \sqrt{\frac{R+\gamma}{4\Delta x}}\xi(x,t)$$

increases diffusion coefficient in KPZ equation

cf. photon condensation model

$$i\frac{\partial\psi}{\partial t} = \frac{-\nabla^2}{2m}\psi + \frac{i}{2}\left(B_{21}(\omega)M_2 - B_{12}(\omega)M_1 - \gamma\right)\psi + \cdots$$



$$\frac{B_{12}(\omega)}{B_{21}(\omega)} = e^{\beta(\omega-\omega_0)} \approx e^{\beta(\omega_c-\omega_0)}[1+\beta(\omega-\omega_c)] \to e^{\beta(\omega_c-\omega_0)}(1+\beta i\frac{\partial}{\partial t})$$

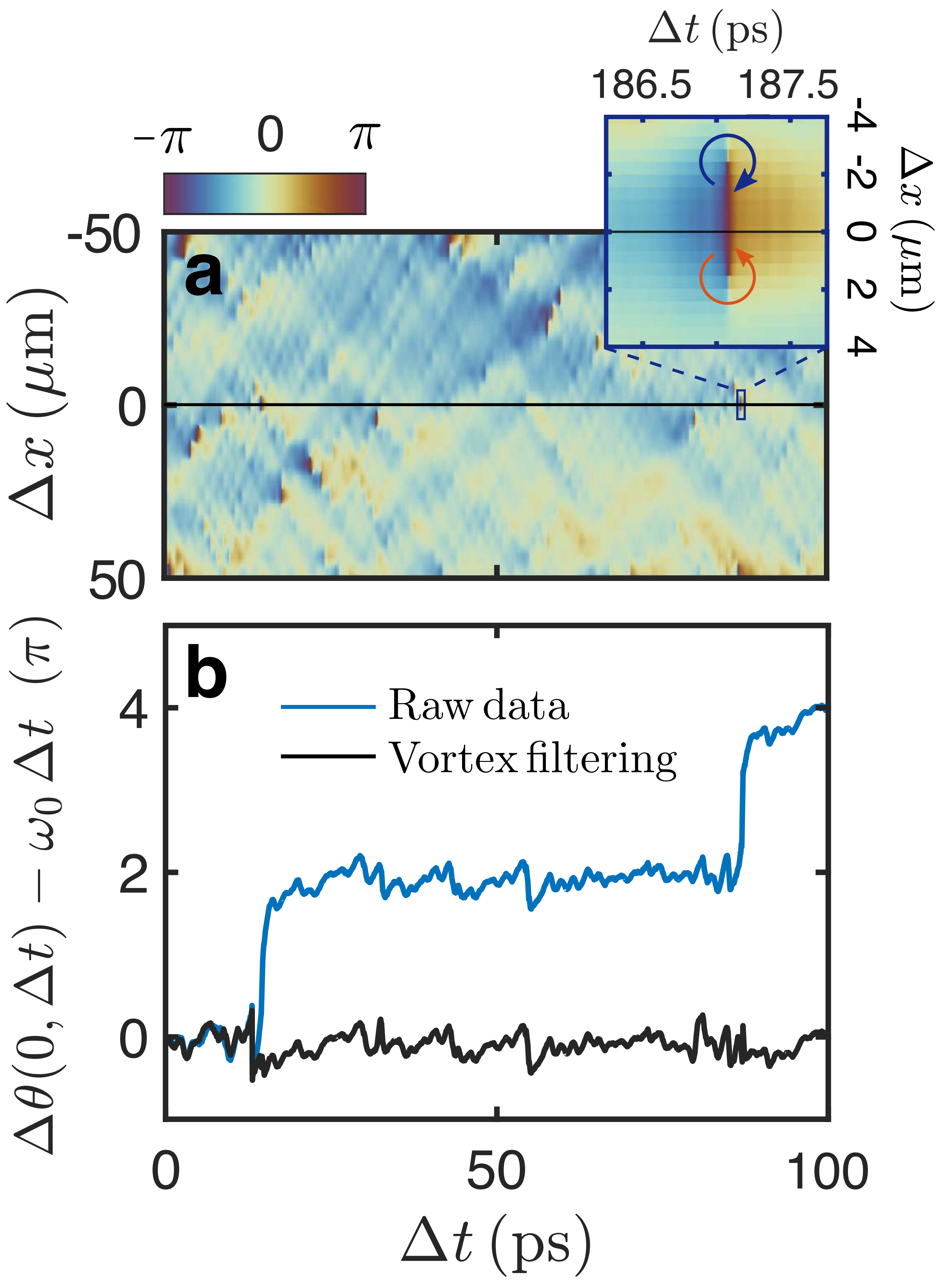
take $B_{21} = B_{21}^c$

and $B_{12}M_1 = B_{12}^c M_1(1+\beta i\frac{\partial}{\partial t}) = B_{12}^c M_1 + i\kappa\frac{\partial}{\partial t}$ with $\kappa = \frac{B_{12}M_1}{2k_BT}$

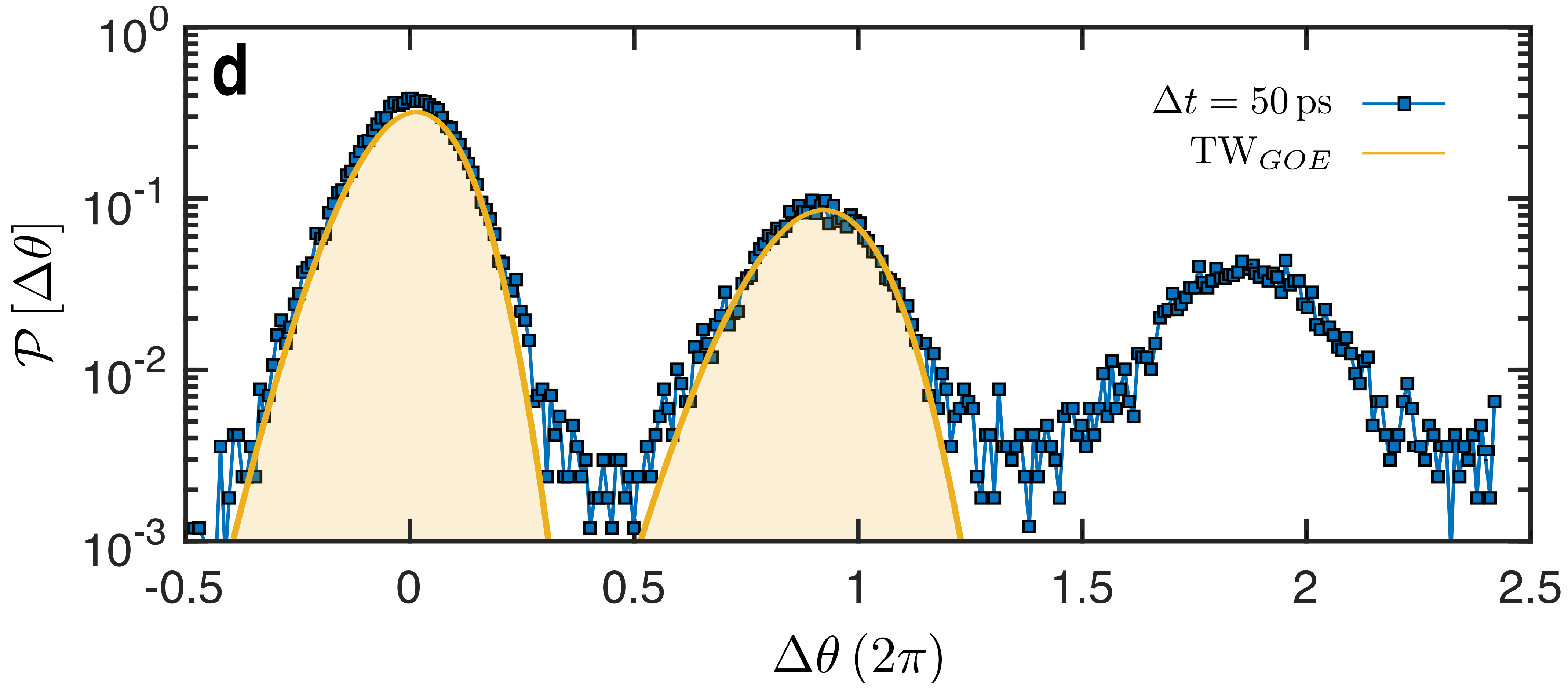
$$i\frac{\partial\psi}{\partial t} = \frac{-\nabla^2}{2m}\psi - i\kappa\frac{-\nabla^2}{2m}\psi + \frac{i}{2}\left(B_{21}^c M_2 - B_{12}^c M_1 - \gamma\right)\psi + \cdots$$

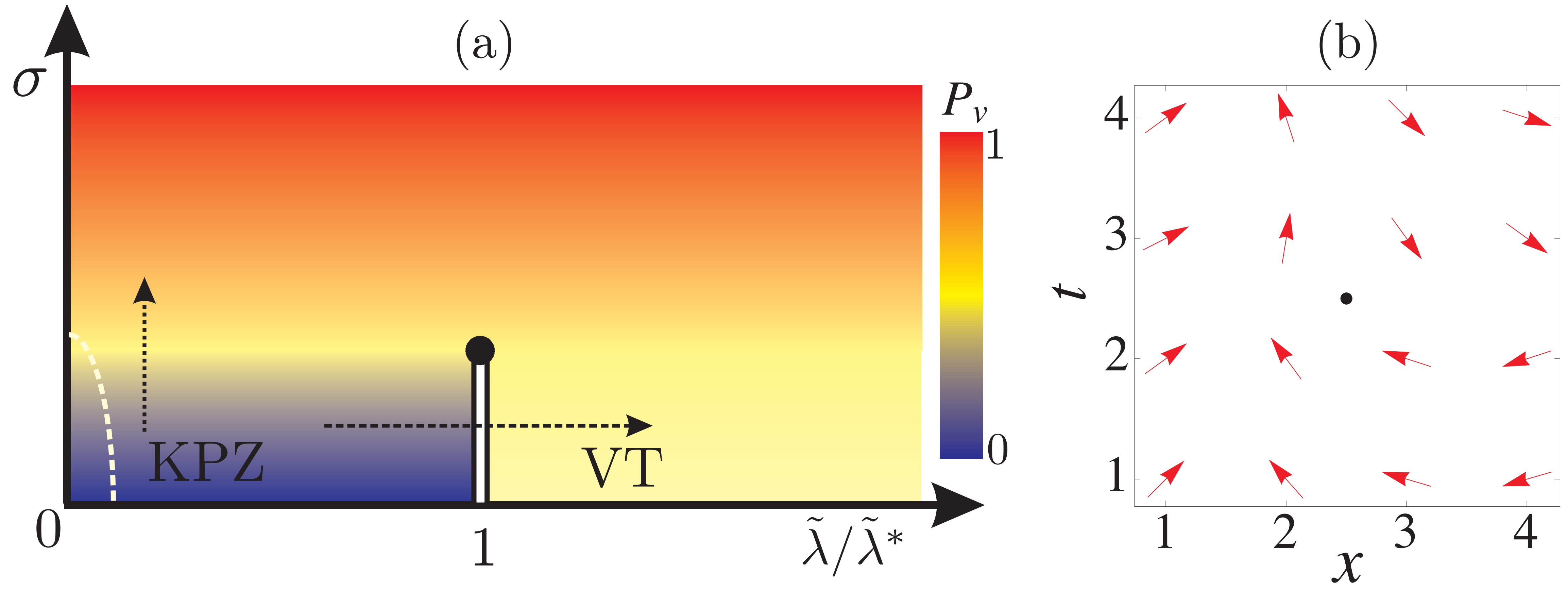
V. N. Gladilin and M.W., Phys. Rev. A 101, 043814 (2020); see also: Chris Toebes, Mario Vretenar, Jan Klaers, arXiv:2110.01898

phase slips (phase-time vortices)



Phase statistics

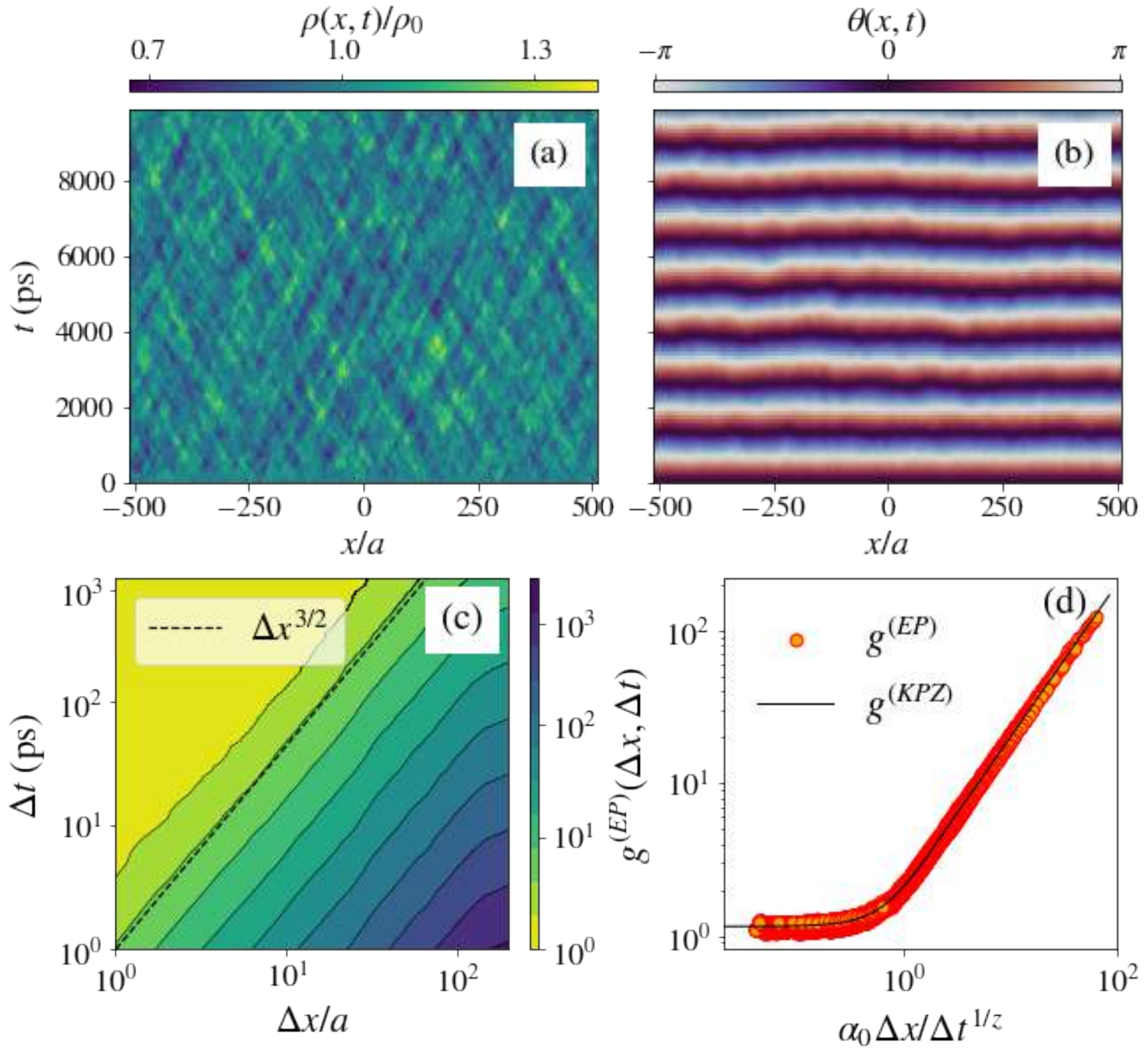




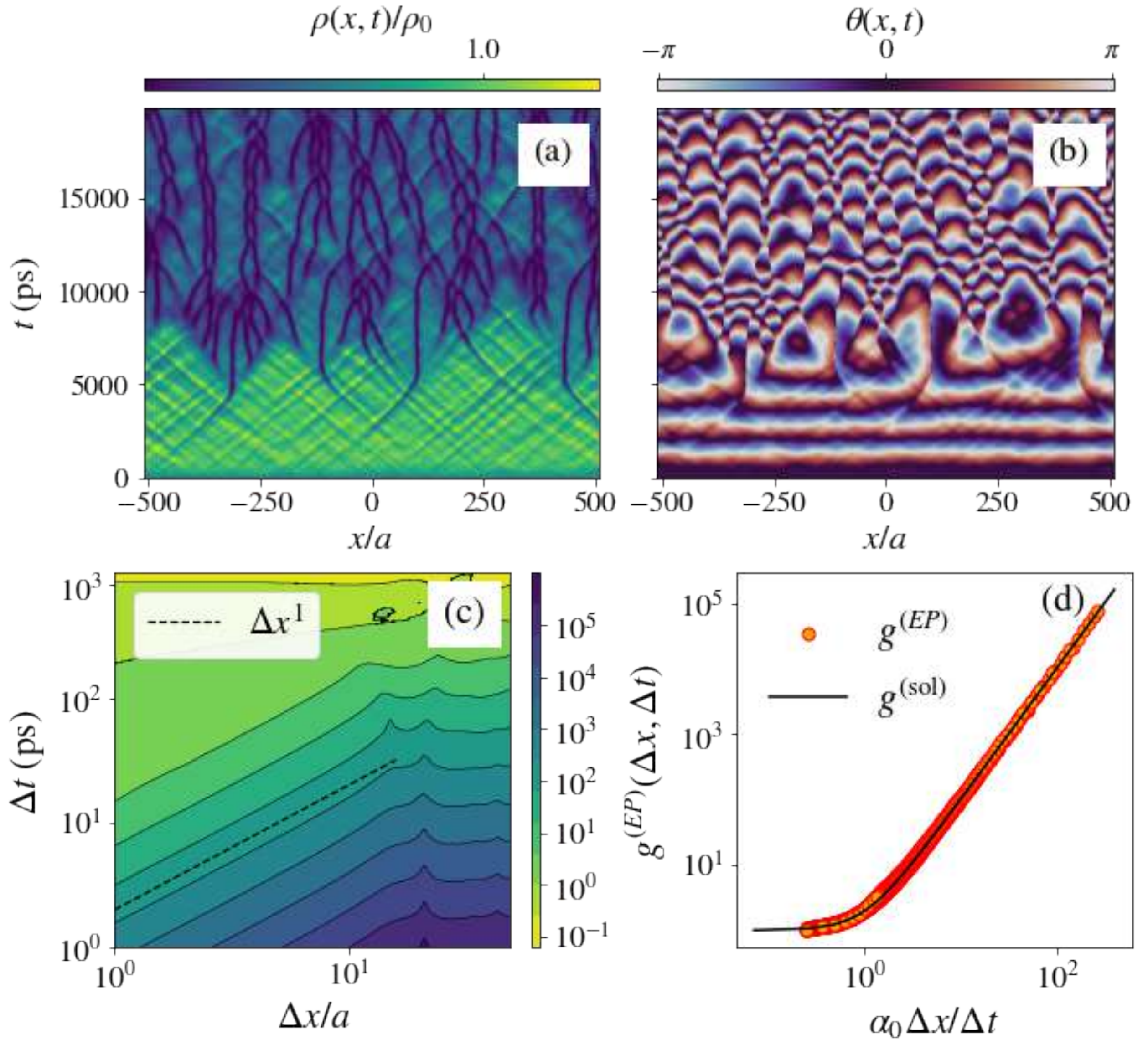
L. He et al. PRL 118 085301 (2017)

More complicated phase dynamics

KPZ

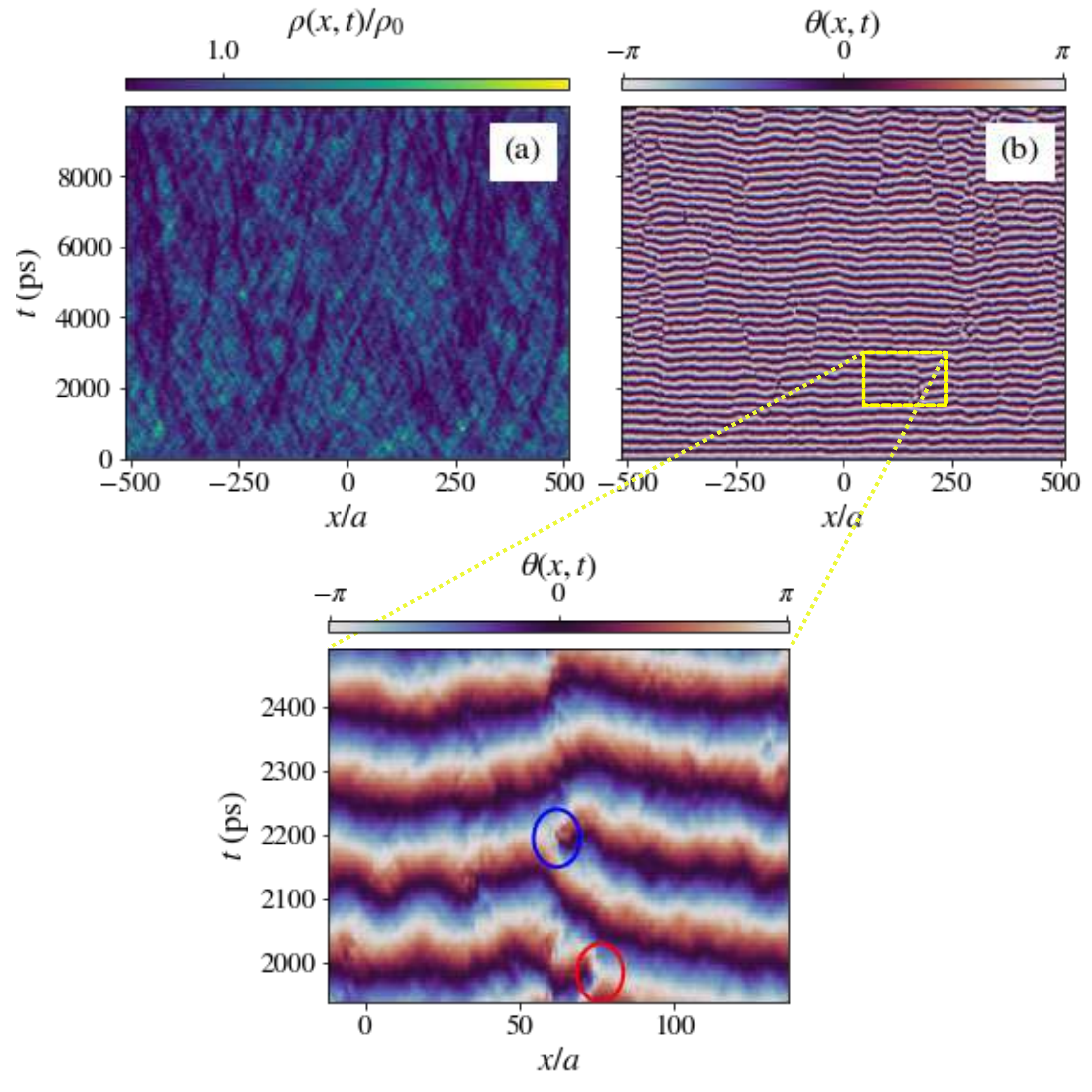


Solitons



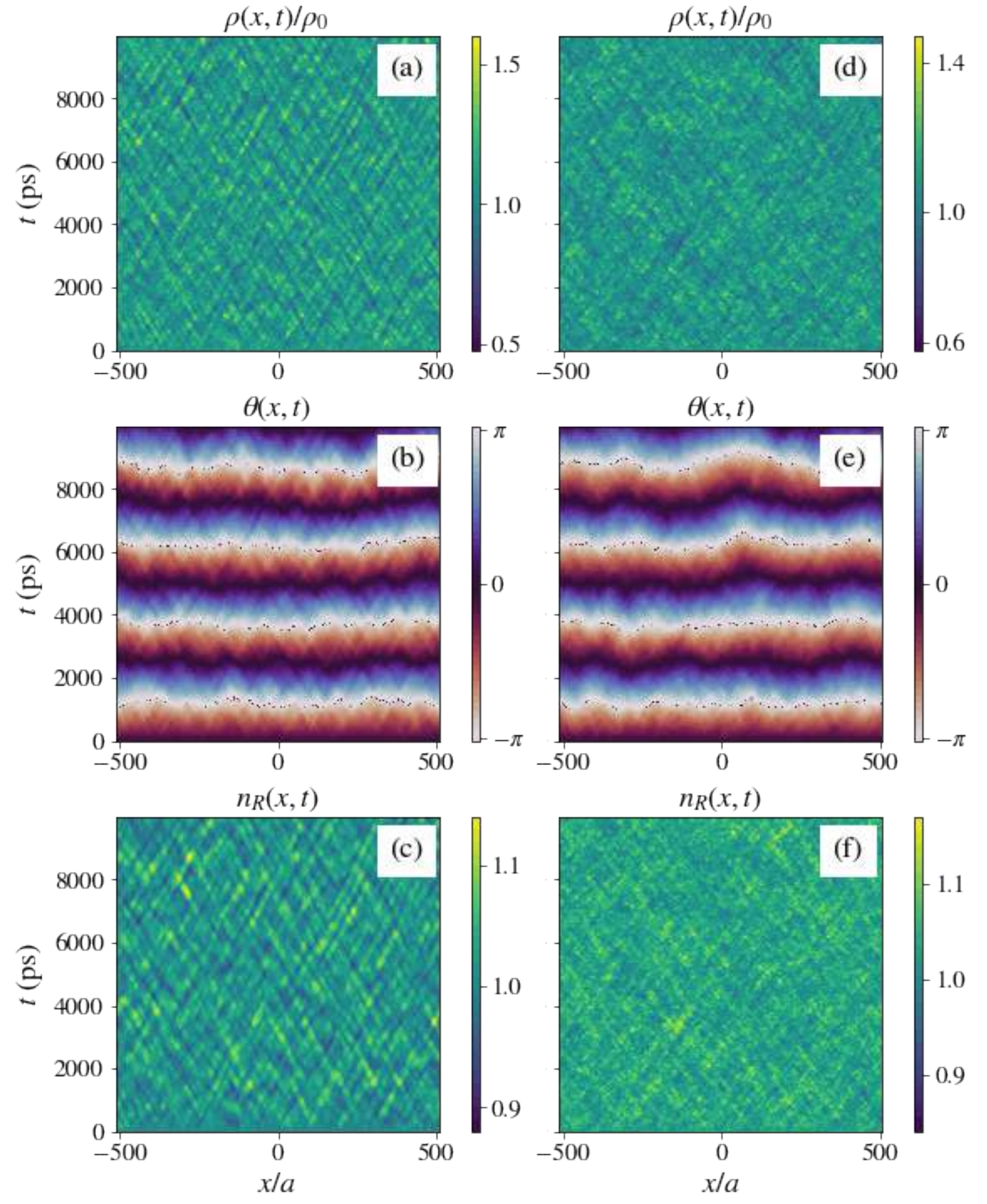
dyn. instability

Space-time vortices



large noise

Reservoir textured



far from adiabatic

F. Vercesi et al, PRR **5** 043062 (2023)

2D

2D Phase correlator in EW (Bog.) approx.

equal time

$$\langle(\theta(x) - \theta(0))^2\rangle \sim \ln|x|$$

$$\Rightarrow \quad \langle\psi^*(x,t)\psi(0,t)\rangle \approx n e^{-\frac{1}{2}\langle[\theta(x,t)-\theta(0,t)]^2\rangle} \sim |x|^{-\alpha}$$

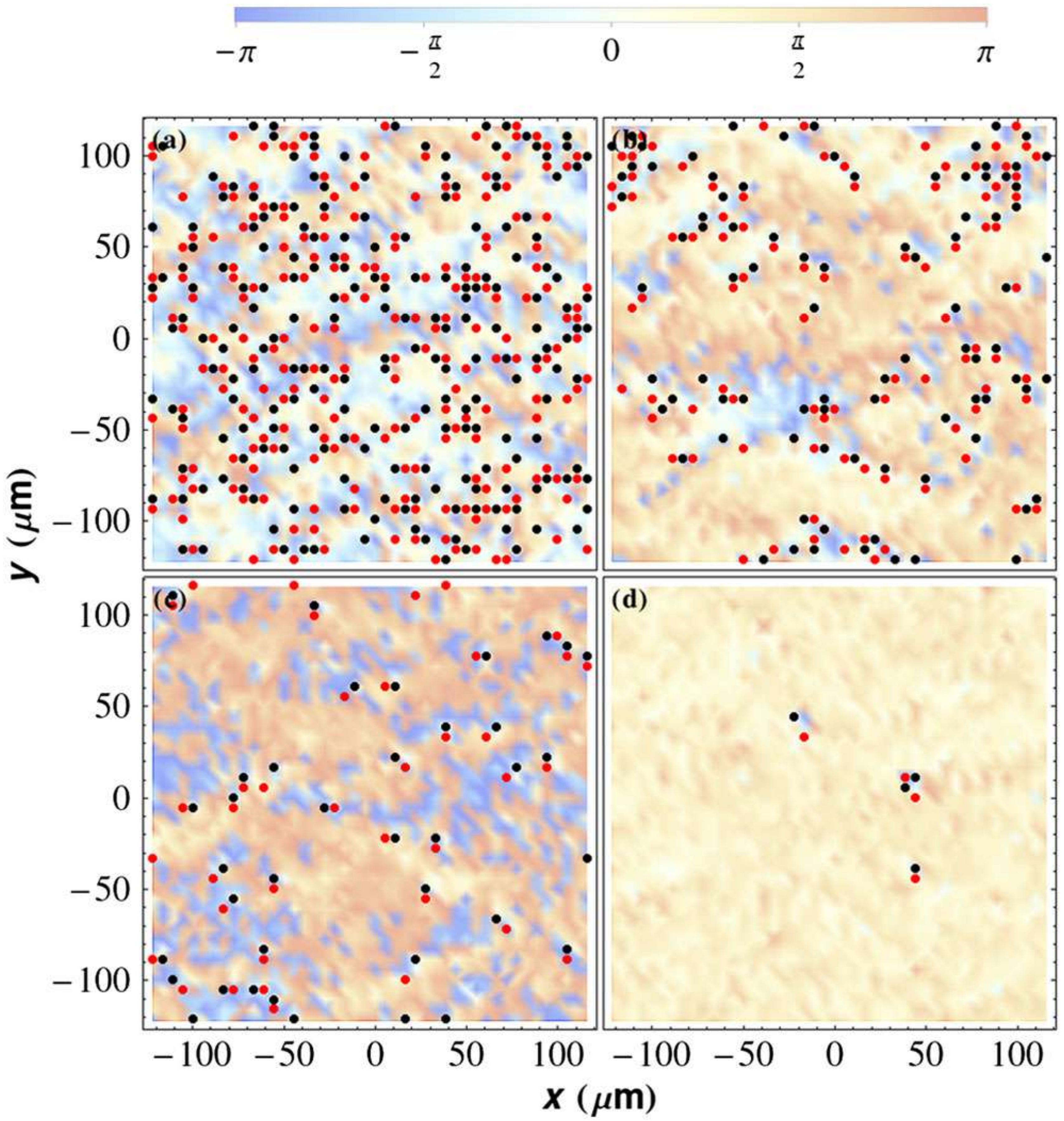
in the absence of vortices

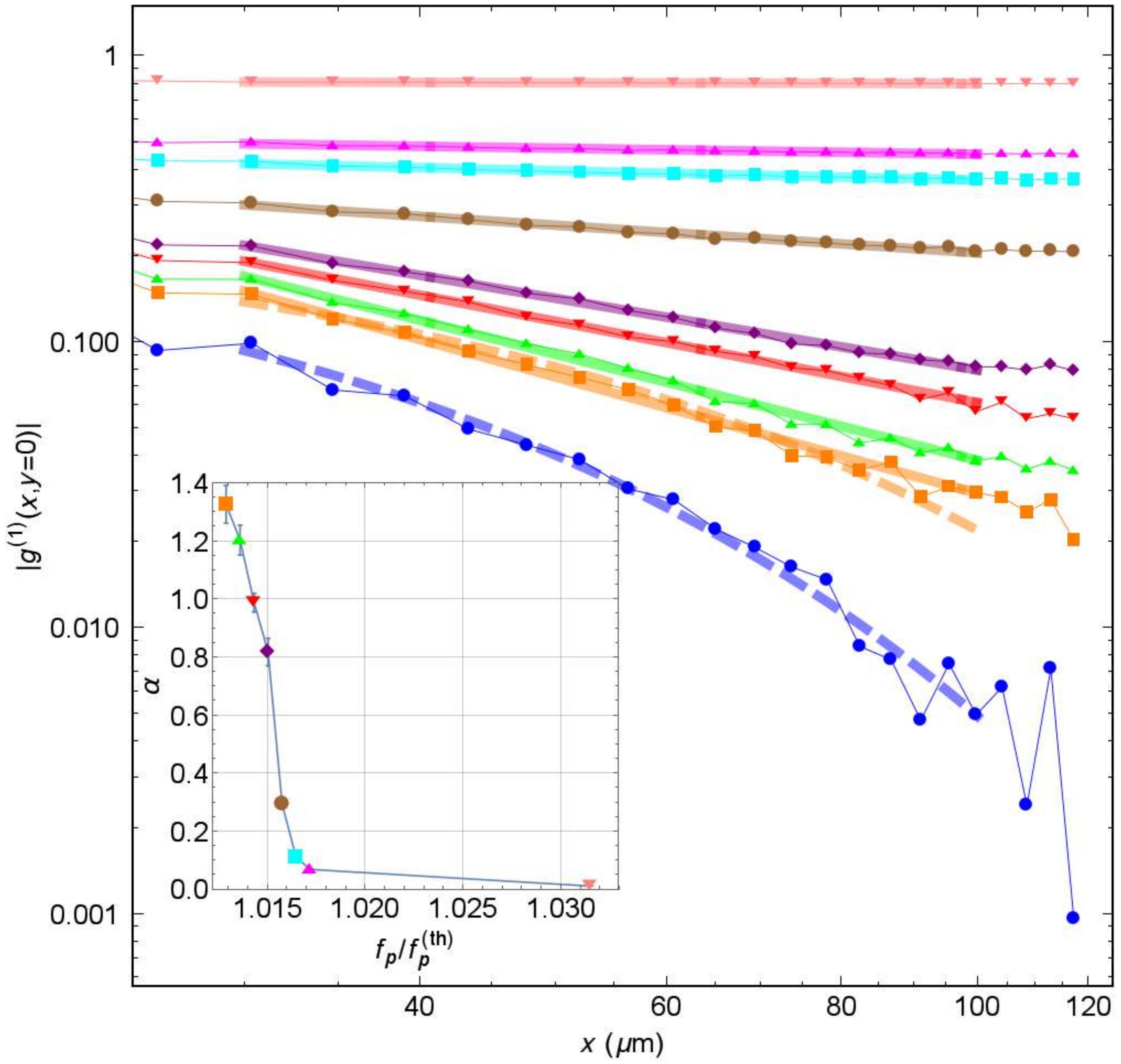
equal space

$$\langle(\theta(x,t) - \theta(x,0))^2\rangle \sim \log(\sqrt{t})$$

$$\Rightarrow \quad \langle\psi^*(x,t)\psi(x,0)\rangle \approx n e^{-\frac{1}{2}\langle[\theta(x,t)-\theta(x,0)]^2\rangle} \sim |t|^{-\alpha/2}$$

noneq BKT





$g^{(1)} \sim r^{-\alpha}$

$g^{(1)} \sim e^{-r/\ell_c}$

G. Dagvadorj et al. PRX 2015

2D KPZ

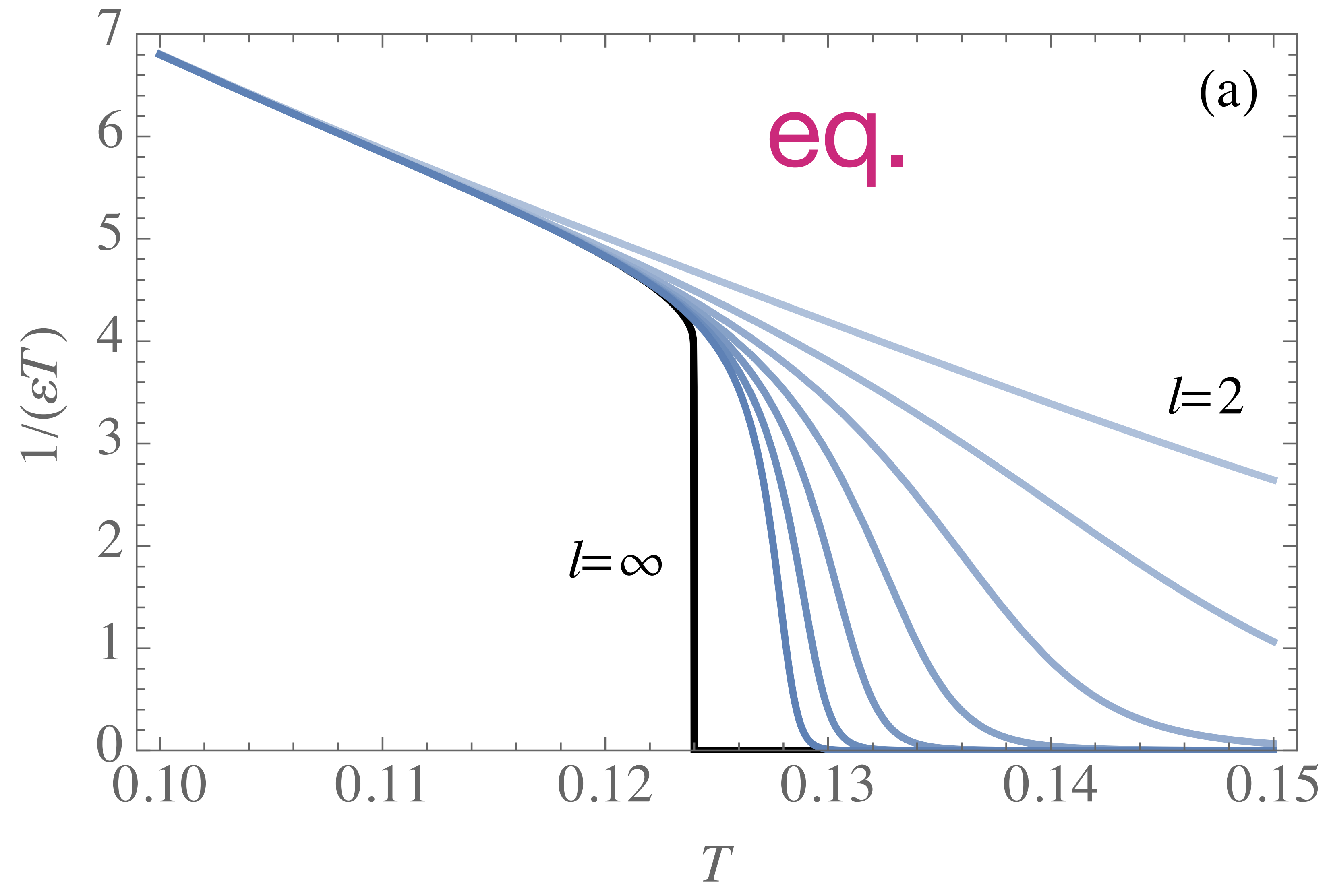
Correlations from power law (equilibrium) to stretched exponential

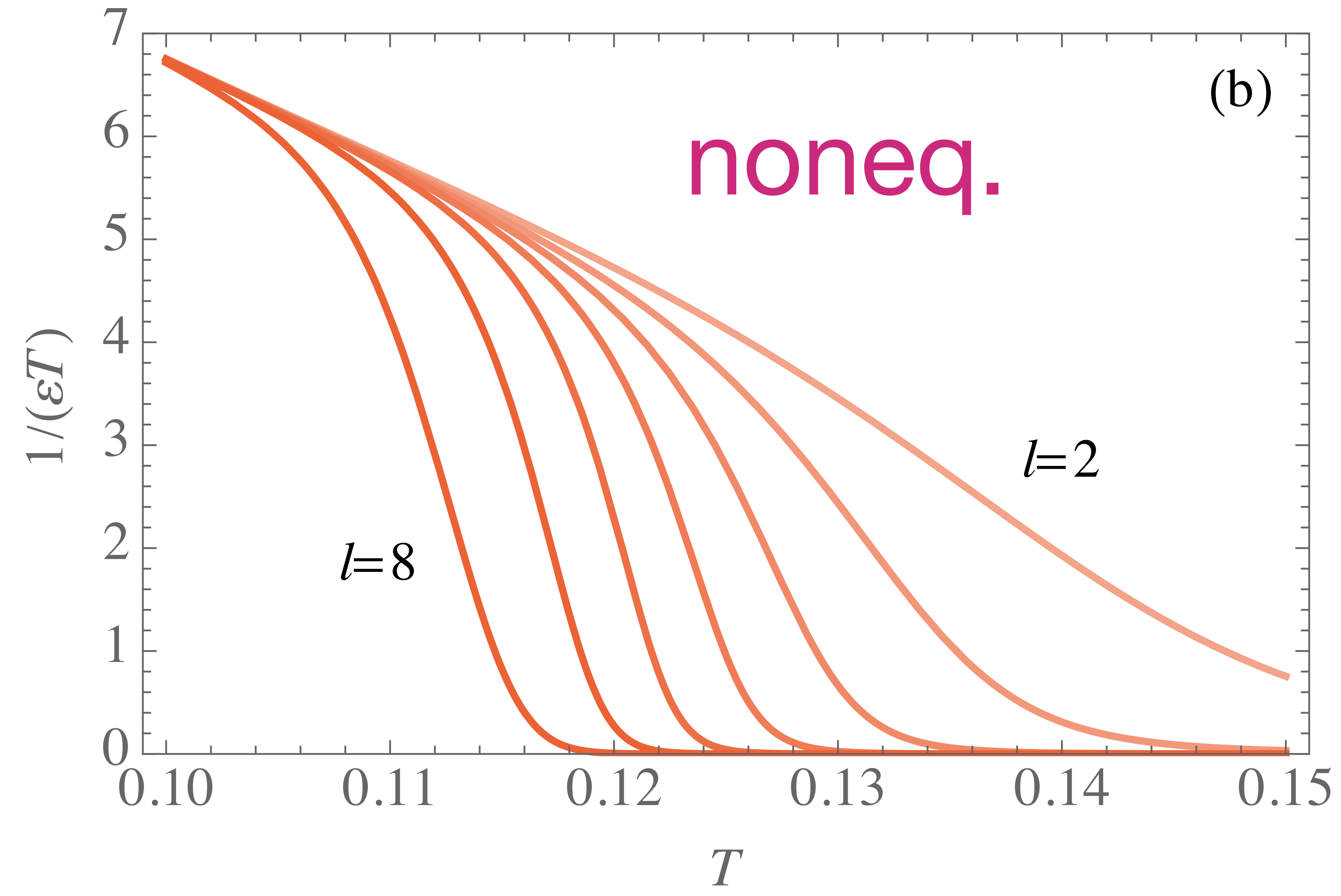
$$g^{(1)}(r) \sim \exp(-r^{\alpha})$$

KPZ: $\alpha \approx 0.39$

$\beta \approx 0.24$

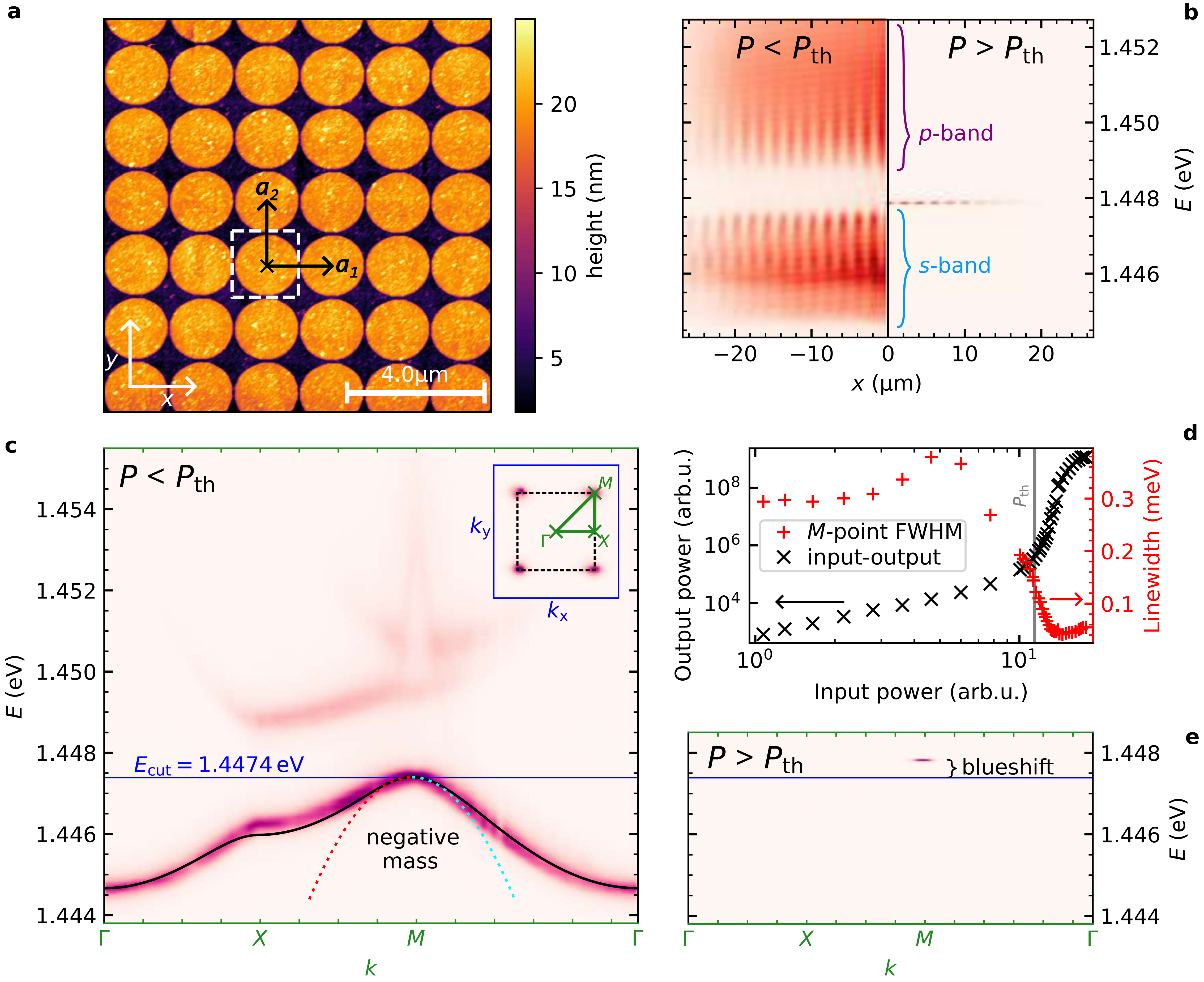
$z \approx 1.61$





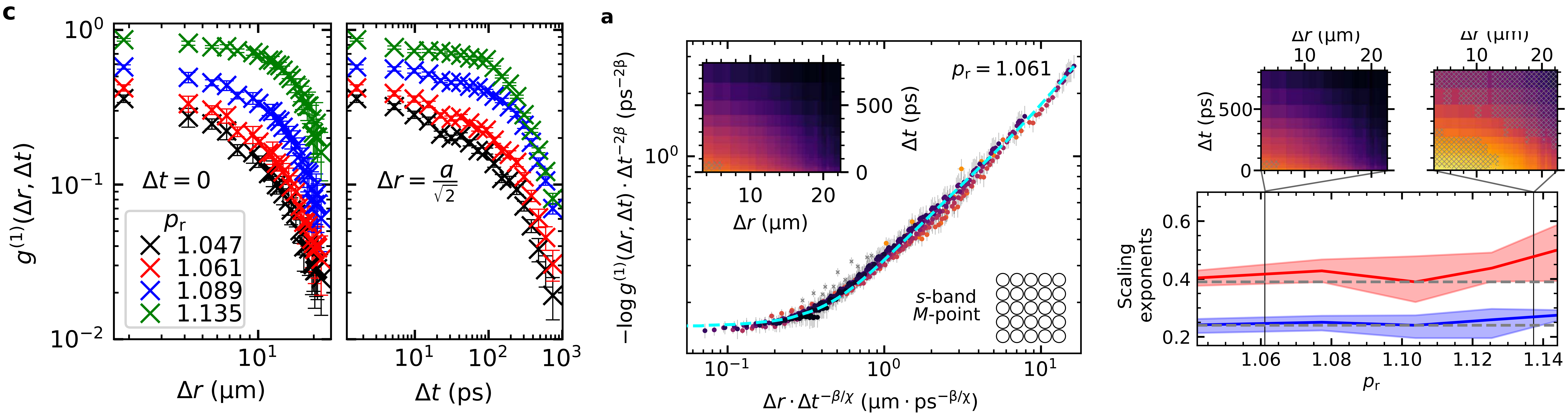
G. Wachtel et al. PRB **94** 104520 (2016)

2D KPZ in polaritons



S. Widmann et al. arxiv:2506.15521

2D KPZ in polaritons



S. Widmann et al. arxiv:2506.15521

Summary

- Condensates of light are well described by stochastic dissipative classical field models

- (Weak) phase fluctuations are in the KPZ universality class, but of of compact variable

- Topics not covered here: nonhermitian physics, topology, superfluidity, analog Hawking, blockade physics

I. Carusotto and C. Ciuti RMP 2013,
I. Carusotto, J. Bloch and MW. Nat. Phys. Rev. 2022