II Collective modes of the superfluid phase from the BCS to the BEC regime

We assume now $T < T_c$ (we restrict to $T=0$)

and we span the crossover $1/k_F a \in]-\infty, +\infty[$

A - BCS approximation of the ground state

The normal ground state $|FS\rangle$ is unstable

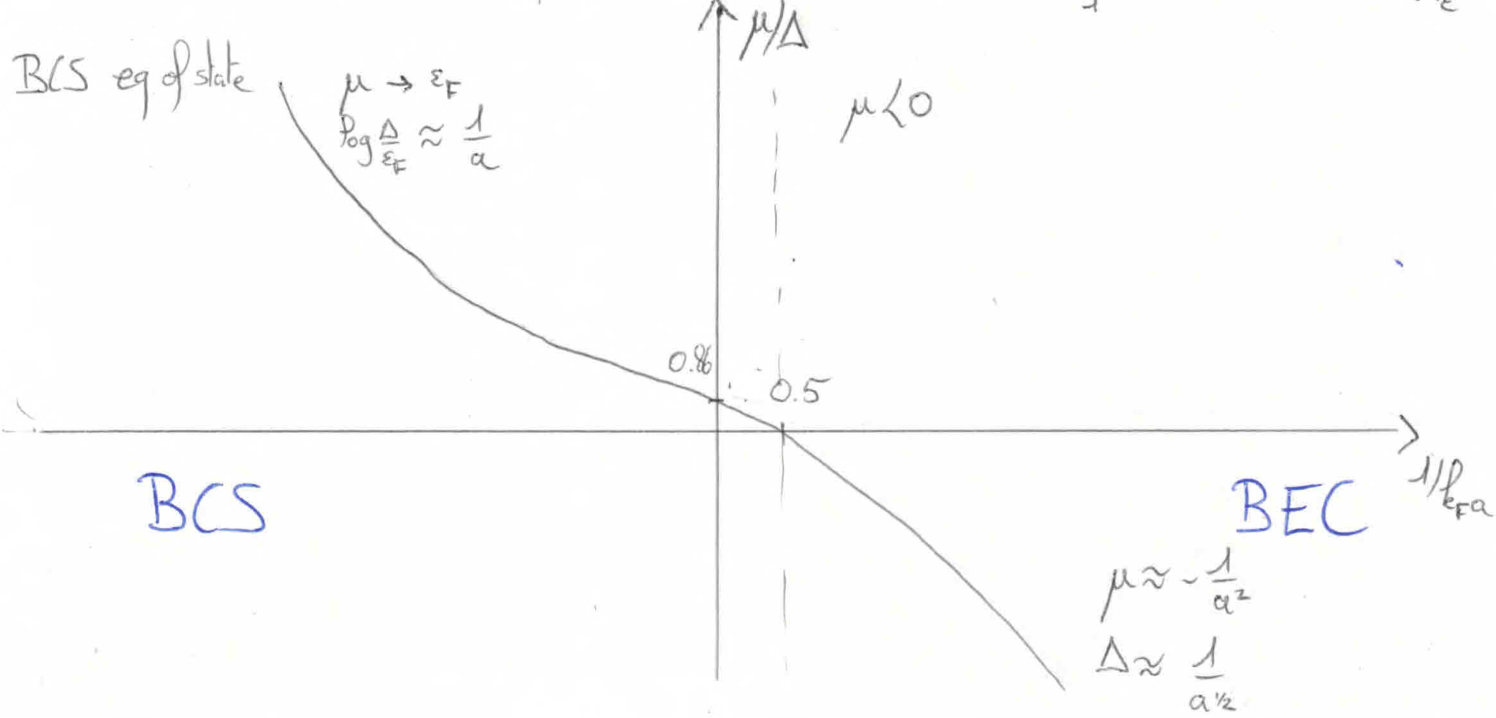
$\Rightarrow$ search the ground state among
$$|\psi_{BCS}\rangle = \mathcal{N} \exp\left(\sum_{\vec{p}} \phi_{\vec{p}}\, a^+_{\vec{p}\uparrow} a^+_{-\vec{p}\downarrow}\right)|0\rangle$$
$$= \prod_{\vec{p}} \left(U_{\vec{p}} - V_{\vec{p}}\, a^+_{\vec{p}\uparrow} a^+_{-\vec{p}\downarrow}\right)|0\rangle$$
$$\left(|U_{\vec{p}}|^2 = 1 - |V_{\vec{p}}|^2\right)$$

Variational principle $\dfrac{\partial E}{\partial V_{\vec{p}}} = 0 \rightarrow V^0_{\vec{p}} = \sqrt{\dfrac{1}{2}\left(1 - \dfrac{\xi_{\vec{p}}}{\sqrt{\xi_{\vec{p}}^2 + \Delta^2}}\right)}$ $\quad \xi_{\vec{p}} = \dfrac{p^2}{2m} - \mu$

$$\begin{cases} \Delta = -\dfrac{g_0}{L^3} \sum_{\vec{p}} U_{\vec{p}} V_{\vec{p}} & \text{gap eq} \\ \rho = 2 \sum_{\vec{p}} V_{\vec{p}}^2 & \text{number eq} \end{cases}$$

⚠ When $\Delta \to 0$, $|\psi_{BCS}\rangle = |FS\rangle_0$ not $|FS\rangle$ → poor description of the normal state at $T \approx T_c$

BCS eq of state



B - Density/pairing response functions

2^{nd} order cumulant approximation for $n_{\vec{p}\sigma} = \langle \hat{a}^+_{\vec{p}+\vec{q}\sigma} a_{\vec{p}\sigma} \rangle$

and $d_{\vec{p}} = \langle \hat{a}_{-\vec{p}-\vec{q}\downarrow} \hat{a}_{\vec{p}\uparrow} \rangle$

($\Longleftrightarrow$ time-dependent Hartree-Fock-Bogoliubov approximation)
"superfluid RPA"

$$i\partial_t \begin{pmatrix} n_{\vec{p}\uparrow} \\ n_{\vec{p}\downarrow} \\ d_{\vec{p}} \\ d^*_{\vec{p}-\vec{q}} \end{pmatrix} = \mathcal{M}_{\vec{p}}(n^{eq},\Delta^{eq}) \begin{pmatrix} n_{\vec{p}\uparrow} \\ n_{\vec{p}\downarrow} \\ d_{\vec{p}} \\ d_{-\vec{p}-\vec{q}} \end{pmatrix} + \mathcal{C}_{\vec{p}}(n^{eq},\Delta^{eq}) \begin{pmatrix} \delta\Delta + U_{pair} \\ \delta\Delta^* + U^*_{pair} \\ \delta\rho_\uparrow + U_\uparrow \\ \delta\rho_\downarrow + U_\downarrow \end{pmatrix}$$

diagonalize $\mathcal{M}$ using BCS qp : $b_{\vec{p}\uparrow} = U_{\vec{p}} a_{\vec{p}\uparrow} + V_{\vec{p}} a^+_{-\vec{p}\downarrow}$

Response matrix

$\delta\Delta = \delta|\Delta| + i\Delta_{eq}\,\delta\varphi$

$$\begin{pmatrix} \delta\varphi \\ \delta|\Delta| \\ \delta\rho \\ \delta s \end{pmatrix} = \left(\begin{array}{c|c} \dfrac{\Pi(\omega,\vec{q})}{\frac{1}{g_0} - \Pi(\omega,\vec{q})} & 0 \\ \hline 0 & \chi_{44}(\omega,\vec{q}) \end{array}\right) \begin{pmatrix} U_{pair} \\ U^*_{pair} \\ U_\uparrow + U_\downarrow \\ U_\uparrow - U_\downarrow \end{pmatrix}$$

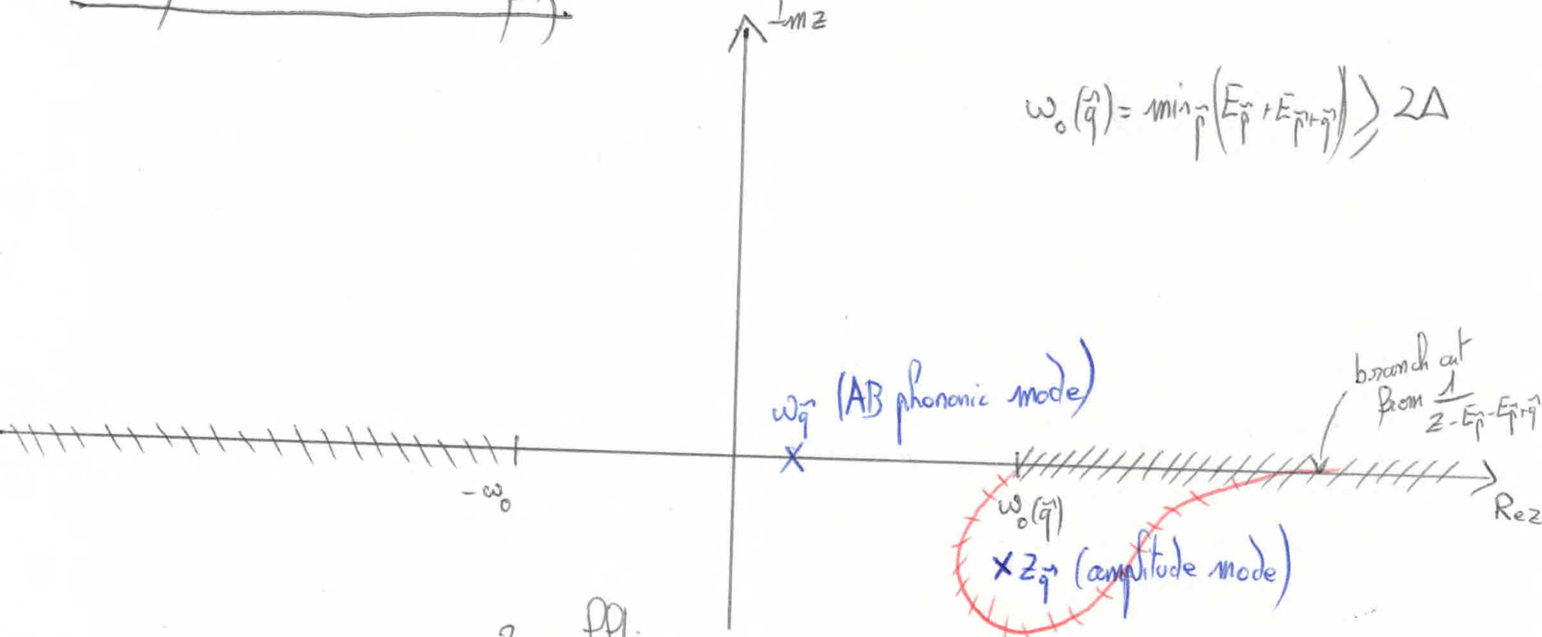
Π: 3x3 matrix

ex
$$\frac{1}{g_0} - \Pi_{\varphi\varphi}(\omega,\vec{q}) = \sum_{\vec{p}} W^2_{\vec{p}\vec{q}} \left[\frac{1}{\omega - E_{\vec{p}} - E_{\vec{p}+\vec{q}}} - \frac{1}{z + E_{\vec{p}} + E_{\vec{p}+\vec{q}}} \right] + \frac{m}{p^2}$$

$W_{\vec{p}\vec{q}} = U_{\vec{p}} U_{\vec{p}+\vec{q}} + V_{\vec{p}} V_{\vec{p}+\vec{q}}$

Analytic structure of χ

$$\omega_0(\vec{q}) = \min_{\vec{p}}\left(E_{\vec{p}} + E_{\vec{p}+\vec{q}}\right) > 2\Delta$$



2 possibilities

$$\det\left(\frac{1}{g_0} - \Pi(z_{\vec{q}}, \vec{q})\right) = 0$$

- $z_{\vec{q}} = \omega_{\vec{q}} \in \mathbb{R}$ undamped mode if $\omega_{\vec{q}} < \omega_0(\vec{q})$
- $\mathrm{Im}\, z_{\vec{q}} < 0$ damped mode in the analytic continuation

C – The Anderson – Bogoliubov phononic branch

At low q

$$\omega_{\vec{q}} = cq\left(1 + \gamma\left(\frac{q}{mc}\right)^2\right) + O(q)^5$$

$mc^2 = \rho \frac{d\mu}{d\rho}$ ($\to$ hydro formula)

$\gamma > 0$ in BEC regime

$\omega_q \to \omega_{\vec{q}}^{Bogo} = \sqrt{c^2 q^2 + \left(\frac{q^2}{2m}\right)^2}$

Bogoliubov branch of the BEC

* $\gamma < 0$ in BCS because $\omega_{\vec{q}} < 2\Delta \ll \varepsilon_F$

(See plots in Appendix)

At high q

* BEC side $(a > 0)$

  $\exists$ a 2-body bound state (a pole of $\frac{1}{g_0} - \Pi$ in vacuum)

  $\Rightarrow \omega_q$ is unbounded

  however $\left|\omega_q - \frac{q^2}{4m}\right| \xrightarrow[q \to 0]{} 0$

  $\frac{q^2}{4m} = \omega_0(q)$, threshold of the scattering continuum

* BCS side $(a < 0)$

  no 2-body bound state

  $\Rightarrow \omega_q$ must disappear at q_{max}

  $q_{max} \approx 2k_F$ in BCS limit

The universal superfluid phononic regime

Phonons are the lowest energy excitations: they completely dominate at $T \ll T_c$
$\omega \ll \Delta$
(in part. since excited qp are suppressed by $n_{\vec{p}}^{eq} \propto e^{-\Delta/T}$)

→ Universal superfluid hydrodynamic: regime of dilute phonons

$$\hat{H}_{eff} = \hat{H}_{hydro}\left(\hat{\rho}, \hat{\vec{v}} = \frac{1}{m}\vec{\nabla}\hat{\phi}\right) = \int d^3r \left(\frac{1}{2} m \hat{\vec{v}}\cdot(\hat{\rho}\hat{\vec{v}}) + e_0(\hat{\rho})\right)$$

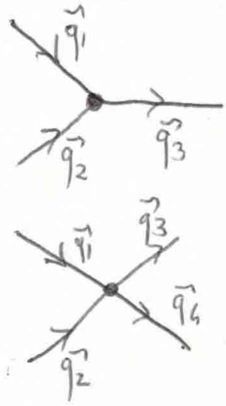
$\hat{\phi}$: phase op. of the condensate

e_0: ground-state energy density

Dissipation drive by phonon interactions:

<u>Phonon-phonon collisions</u>

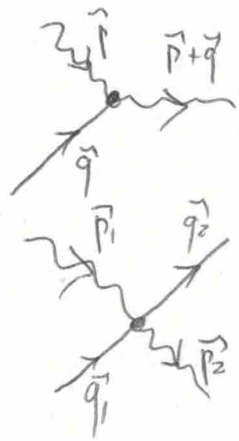
set the transport properties



(only if $\gamma > 0$)

<u>Phonon-quasiparticle collisions</u>

set the quasiparticle lifetime

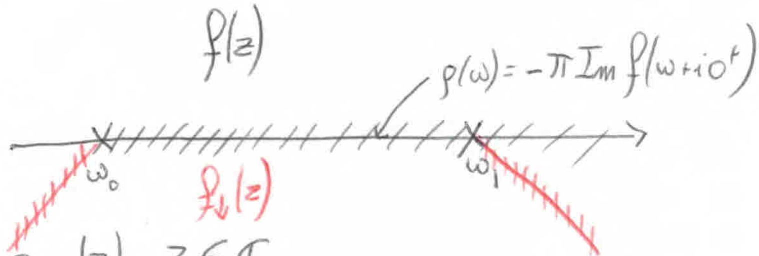


On the BEC side the thermal population of the AB branch depopulates the condensate and drive the superfluid to normal transition

D- The amplitude branch in the pair-breaking continuum

* Analytic continuation Let $f(z) = \sum_{\vec{p}} \frac{W_{\vec{p}}}{z - 2E_{\vec{p}}}$

→ compute the spectral density $\rho(\omega) = \sum_{\vec{p}} W_{\vec{p}}\, \delta(\omega - 2E_{\vec{p}})$
(usually easier to calc than f)



→ Extend ρ to $\mathbb{C}$: $\rho(\omega) \longrightarrow \rho_{ext}(z),\ z \in \mathbb{C}$

→ Analytic continuation of f to $\mathrm{Im}\, z < 0$

$$f_\downarrow(z) = f(z) - 2i\pi \rho_{ext}(z)\ \Theta(-\mathrm{Im}\, z)$$

If $\exists$ multiple branching points, $\exists$ multiple continuations

* Application to the amplitude response

$$\frac{1}{g_0} - \Pi_\downarrow(z_{\vec{q}}, \vec{q}) = 0 \iff z_q = 2\Delta\left(1 + \alpha (q\xi_{pair})^2 + \dots\right)$$

$\alpha = 0.24 - 0.30i$, $\xi_{pair} = \frac{\mu}{4m\Delta^2}$ pair size

Amplitude mode? In the BCS limit $\Pi_{\varphi|\Delta|} \to 0$, thus $\frac{1}{g_0} - \Pi_\downarrow = 0 \iff \frac{1}{g_0} - \Pi_{|\Delta||\Delta|} = 0$

Elsewhere z_q is affected by phase-amplitude mixing
but the spectral weight of the mode remains larger in $\chi_{|\Delta||\Delta|}$

This amplitude mode disappear in the BEC regime and at high q

⚠ Special case of $q=0$
(Experiments in Swinburne and Bonn)

$z_q \xrightarrow[q\to 0]{} 2\Delta$
but $Z_q \to 0$ (residue)

$\chi_{|\Delta||\Delta|}(\omega, q \neq 0) \simeq \frac{Z_q}{(\omega - \mathrm{Re}\, z_q)^2 + (\mathrm{Im}\, z_q)^2}$ Lorentzian ⟺ exponential decay

$\chi_{|\Delta||\Delta|}(\omega, q = 0) \simeq \frac{A}{\sqrt{\omega - 2\Delta}}$ Not a pole ⟹ power-law decay

E – The "preformed pair" / normal Bose fluid regime

* Fluctuations in the bosonic modes drive the SF → normal phase transition in the BEC limit

→ Phase coherence is lost before the pairs break into 2 fermions

Opens a regime of normal Bose fluid at

$$T_c < T \ll T_{pair} \approx \frac{1}{a^2}$$

* Nozières - Schmitt Rink description of preformed pairs at T_c

Keep the BCS-approx Thouless criterion $\displaystyle \frac{1}{g} - \sum_{\vec{p}} \left(\frac{1-2n^{eq}_{\vec{p}}}{2\xi_{\vec{p}}} - \frac{m}{p^2} \right)_{T=T_c} = 0$

but update the number equation to account for preformed pairs

$$\rho = \underbrace{\rho_F}_{\sum_{\vec{p}\sigma} n^{eq}_{\vec{p}}(T)} + 2\underbrace{\rho_B}_{\propto -\sum_{\vec{q}} \frac{\partial \Omega(q)}{\partial \mu}}$$

with $\Omega(q)$ the contribution of pair fluctuations of momentum $\vec{q}$ to the Grand Potential Ω

Excitation spectra of the T=0 superfluid phase

Appendix