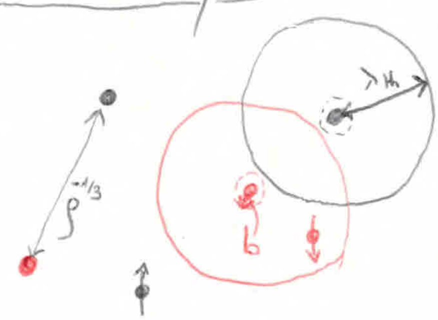
The BCS to BEC crossover from the normal to the superfluid phase

H. Kurkjian, LPTMC, Sorbonne Université, CNRS
(hadrien.kurkjian@cnrs.fr)

Introduction

* Considered system



- Gas of atomic fermions in 2 internal states ($\uparrow$ and $\downarrow$ by convention)
- 3D (possibly also 2D)
- Short-range interactions between $\uparrow$ and $\downarrow$

$$b \ll \rho^{-1/3} \Rightarrow \text{dilute gas}$$

range of the potential; interp. distance (ρ = total density)

- Low energy / temperature $\Rightarrow$ cold gas

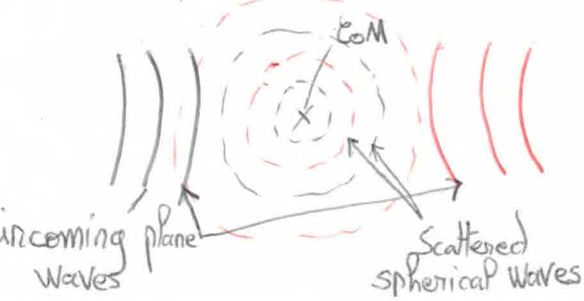
$$b \ll \lambda_{th} = \sqrt{\frac{2\pi}{kT}}$$

but $\rho^{1/3}\lambda_{th} \approx \frac{T}{T_F} \approx 1 \rightarrow$ Classical ($T \gg T_F$) to quantum degenerate ($T \ll T_F$) crossover

Fermi temperature

* Effective model of the interactions

2-body scattering in the cold-dilute regime looks like



$$\psi_i = \psi_0 e^{i\vec{k}_i\cdot\vec{r}} \quad \text{(incident wave function)}$$

$$\psi_s(\vec{r}) = \boxed{a}\,\psi_0 \frac{e^{ik_i r}}{r} \quad \text{(s-wave scattered wave)}$$

a : the scattering length is set by the true potential (Van der Waals interactions) 2

Since it is the only relevant parameter of the 2-body problem, we can replace the potential by

$$\hat{V} = g_0 \int d^3r\, \psi_\uparrow^\dagger(\vec{r})\, \psi_\downarrow^\dagger(\vec{r})\, \psi_\downarrow(\vec{r})\, \psi_\uparrow(\vec{r})$$

→ contact interactions with g_0 adjusted to reproduce the physical parameter a.

$$\frac{1}{g} = \frac{1}{g_0} + \int_{]-\frac{\pi}{\ell}, \frac{\pi}{\ell}]^3} d^3k\, \frac{m}{k^2}$$

$\left(g = \frac{4\pi\hbar^2 a}{m}\right)$

ℓ attice step of discretized space → UV cutoff

→ $a \gg \ell$, $g_0 \underset{\ell\to 0}{\longrightarrow} 0$, $g = \dfrac{1}{\frac{1}{g_0} + \int_{\mathbb{R}^3} d^3k \frac{m}{k^2}}$

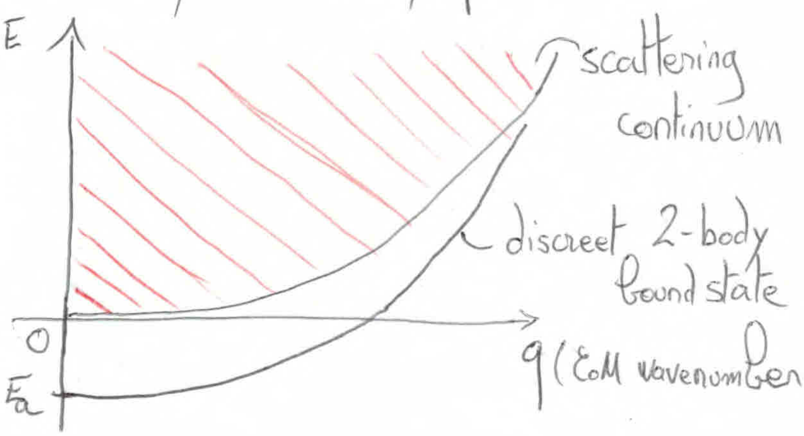
$\ell \to 0$ either → strongly-interacting/resonant regime

→ $k_F \gg \ell \gg a$, $g_0 \underset{g\to 0}{\longrightarrow} 0$

$$g = g_0 + g_0^2 \int_{FBZ} d^3k\, \frac{m}{k^2} + \dots$$

→ weakly-interacting regime

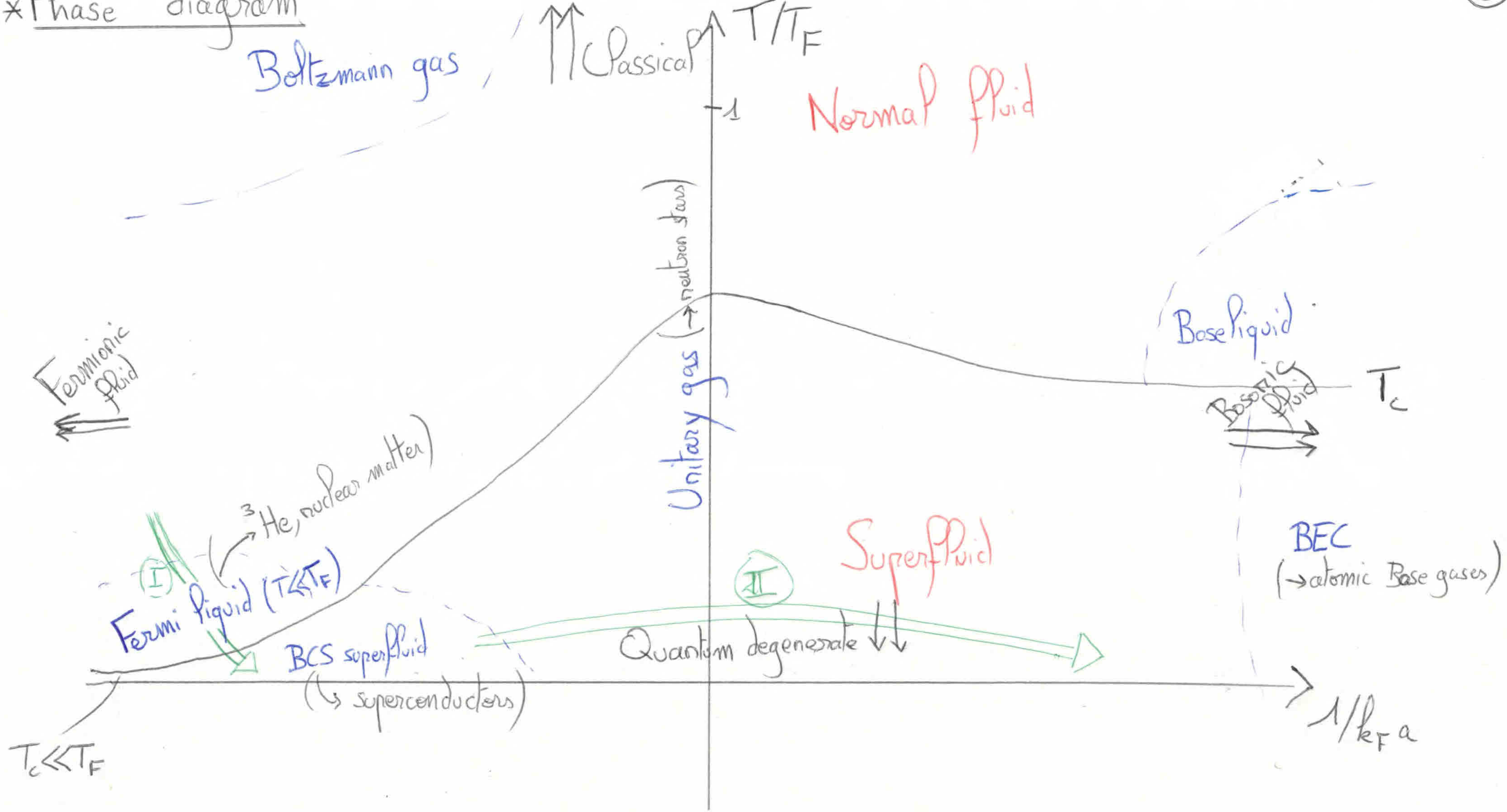
* Solution of the 2-body pbm



2-body BS with $E_a = -\dfrac{\hbar^2}{2ma^2}$ if $a > 0$

behave as a bosonic dimer at energies $E \ll |E_a|$

* Phase diagram



Lecture I: Transport and phase transition with Landau qp

II: Collective modes of the superfluid

$k_F = (3\pi^2 \rho)^{1/3}$, $k_B T_F = \frac{\hbar^2 k_F^2}{2m}$

I Quasiparticles from the Fermi liquid regime to the BCS superfluid

Fermi liquid theory : 60's theory developped for ^3He
later applied to electron gases / nuclear matter

We will use it to obtain an exact description of the gas (static or dynamic)
at $\boxed{T_c \ll T \ll T_F}$

Since $T_c/T_F \approx e^{-\frac{\pi}{2(k_F|a|)}}$, this limits us to $1/k_F a \lesssim -1$
weakly to moderate interaction regime
not unitarity

A- From particle to quasiparticle states

* Hamiltonian: $\hat{H} = \hat{H}_0 + \hat{V}$ with $\hat{H}_0 = \sum_{\vec{p},\sigma=\uparrow\downarrow} \frac{p^2}{2m} \hat{a}^+_{\vec{p}\sigma} \hat{a}_{\vec{p}\sigma}$

  $\hat{a}$: particle annihilation operator

* Eigenbasis of $\hat{H}_0$: the Fock states $|\{n_{\vec{p}\sigma}\}_{\vec{p},\sigma=\uparrow\downarrow}\rangle_0$ of the ideal gas

$$ {}_0\langle\{n_{\vec{p}'\sigma'}\}|\hat{a}^+_{\vec{p}\sigma} a_{\vec{p}\sigma}|\{n_{\vec{p}'\sigma'}\}\rangle_0 = n_{\vec{p}\sigma} $$

few examples: $|FS\rangle_0 = |\{n_{\vec{p}\sigma} = \Theta(p_F - p)\}\rangle_0$: ground state of $\hat{H}_0$

$(p_F = \hbar k_F)$

$|\vec{p}\sigma\rangle_0 = \hat{a}^+_{\vec{p}\sigma}|FS\rangle_0$ for $p > p_F$

$|\vec{p}\sigma\rangle_0 = \hat{a}_{p\sigma}|FS\rangle_0$ for $p < p_F$

* The adiabatic dressing hypothesis

If Landau's quasiparticle picture is correct, there must exist a state $|\{n_{\vec{p}\sigma}\}\rangle$ adiabatically connected to $|\{n_{\vec{p}\sigma}\}\rangle_0$ as $\hat{V} \to 0$

expansion in powers of $\hat{V}$: $|\{n_{\vec{p}\sigma}\}\rangle = |\{n_{\vec{p}\sigma}\}\rangle_0 + |\{n_{\vec{p}\sigma}\}\rangle_1 + \dots$

Two conceptual problems:

1. The quasiparticle states $|\{n_{\vec{p}\sigma}\}\rangle$ <u>cannot</u> be the eigenstates of $\hat{H}$

By the Eigenstate Thermalization Hypothesis (ETH), all eigenstates of $\hat{H}$ are equivalents for non-microscopic observables.

States $|\vec{p}\sigma\rangle$ and $|\vec{p}'\sigma\rangle$ with $p = p'$ have the same energy but are not equivalent for e.g. the momentum distribution.

Qp states are then <u>not ergodic</u>: they must decay

$\Rightarrow$ quasiparticle lifetime

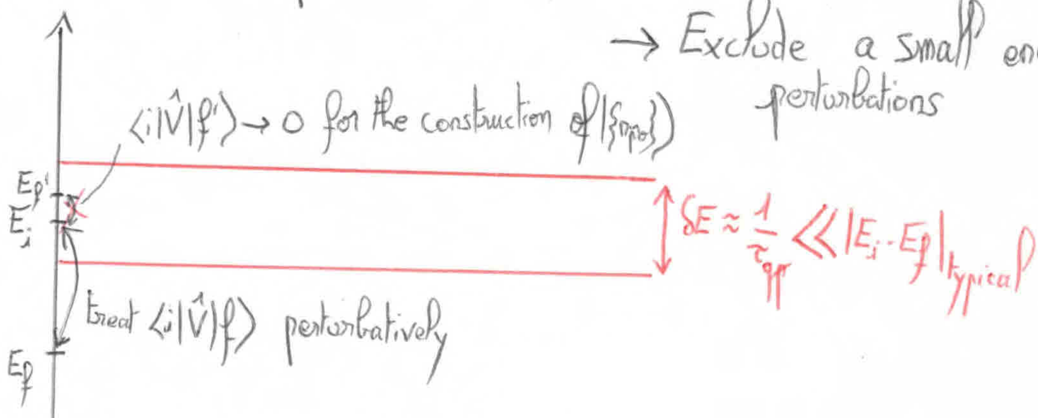
2. States $|\{n_{\vec{p}\sigma}\}\rangle_0$ form a highly-degenerate continuum

$\rightarrow$ Perturbation theory fails for $|i\rangle, |f\rangle$ when $|E_i - E_f| < |\langle\hat{V}\rangle|$

$\Rightarrow$ no hope to construct the $|\{n_{\vec{p}\sigma}\}\rangle$ with standard perturbation theory

* What the qp states really are

→ Exclude a small energy window (δE) from perturbations



In this way $|\{n_{\vec{p}\sigma}\}\rangle$ are related to $|\{n_{\vec{p}\sigma}\}\rangle_0$ by a unitary transform

$$|\{n_{\vec{p}\sigma}\}\rangle = e^{\hat{S}}|\{n_{\vec{p}\sigma}\}\rangle_0 \quad \text{with } S^+ = -S$$

$$e^{\hat{S}} = 1 + \hat{Q}(E)\,\hat{G}(E)\,\hat{V}$$

projector excluding states with $E_f \in \left[E - \frac{\delta E}{2}, E + \frac{\delta E}{2}\right]$

$\hat{G}(z) = \dfrac{1}{z - \hat{H}}$ resolvent ($\approx$ Green's function)

ex to first order in $\hat{V}$

$$|i\rangle = |i\rangle_0 + \sum_{|f\rangle_0} \frac{{}_0\langle f|\hat{Q}(E_i^0)\hat{V}|i\rangle_0}{E_f^0 - E_i^0}|f\rangle_0$$

The price to pay: we will have to treat on-shell couplings (exactly) later on

B- Eigenenergy, interaction fonctions and residu of the qp

We need to express $\hat{H}$ in the new $|\{n_{\vec p\sigma}\}\rangle$ basis.

First: diagonal elements $\langle i|\hat{H}|i\rangle$

Eigenenergy

$$\varepsilon_{\vec p\sigma}[\Omega] \equiv \langle \vec p\sigma, \Omega|\hat{H}|\vec p\sigma, \Omega\rangle - \langle \bar{\vec p\sigma}, \Omega|\hat{H}|\bar{\vec p\sigma}, \Omega\rangle$$

$$n_{\vec p\sigma} = \begin{cases} 0 & \text{in } |\bar{\vec p\sigma}, \Omega\rangle, \\ 1 & \text{in } |\vec p\sigma, \Omega\rangle \end{cases} \qquad \Omega = \{n_{\vec p'\sigma'}\}_{\vec p'\sigma' \neq \vec p\sigma}$$

$\varepsilon_{\vec p\sigma}$ is a functional of Ω (the state of all the qp except $\vec p\sigma$)

eigenergy in the ground state $\varepsilon^0_{\vec p\sigma}$ for $\Omega = \{n_{\vec p'\sigma'} = \Theta(p_F - p')\}_{\vec p'\sigma' \neq \vec p\sigma}$

= "quasiparticle Fermi sea"

near the Fermi level: $\varepsilon^0_{\vec p\sigma} - \mu = \frac{p - p_F}{m^*}$

⚠ $\varepsilon_{\vec p\sigma}$ depends on Ω ⇒ qp states are not Fock states

Interaction fonction

decompose the energy of the 2qp state

$$E_{|\vec p\sigma, \vec p'\sigma', \Omega\rangle} = E_{|\bar{\vec p\sigma}, \bar{\vec p'\sigma'}, \Omega\rangle} + \varepsilon_{\vec p\sigma}\left[\{n_{\vec p'\sigma'} = 0, \Omega\}\right] + \varepsilon_{\vec p'\sigma'}\left[\{n_{\vec p\sigma} = 0, \Omega\}\right] + \frac{1}{V} f_{\vec p\sigma, \vec p'\sigma'}[\Omega] \qquad \left(\Omega = \{n_{\vec p''\sigma''}\}_{\vec p''\sigma'' \neq \vec p\sigma, \vec p'\sigma'}\right)$$

"$f_{\vec p\sigma, \vec p'\sigma'} = \frac{1}{V}\frac{\partial^2 E}{\partial n_{\vec p\sigma}\partial n_{\vec p'\sigma'}}$". In practice $f_{\vec p\sigma, \vec p'\sigma'} \approx f_{\vec p\sigma, \vec p'\sigma'}[\Omega_{FS}]$

⟶ $\varepsilon^0_{\vec p\sigma}$ and $f_{\vec p\sigma, \vec p'\sigma'}$ are enough to characterize the equilibrium p$^{\text{ties}}$

Such as the ground state energy, or the thermodynamic quantities ⑧

C_V: specific heat

κ: compressibility

χ_S: spin susceptibility

<u>Residue</u> $$Z_{\vec p\sigma}[\Omega] = \langle \vec p\sigma,\Omega|\hat a^{+}_{\vec p\sigma}\hat a_{\vec p\sigma}|\vec p\sigma,\Omega\rangle - \langle \vec p\sigma,\Omega|\hat a^{+}_{\vec p\sigma}\hat a_{\vec p\sigma}|\vec p\sigma,\Omega\rangle$$

qp state   p number

breakdown of adiabatic hypothesis when $Z_{\vec p\sigma}=0$

C - <u>Collision amplitudes</u>

Off-diagonal elements of $\hat H$: $\mathcal{A}_{i\to f} = \langle f|\hat H|i\rangle \neq 0$ when $|E_f - E_i| < \delta E$

for $2\leftrightarrow 2$ collisions $$\frac{1}{V}\mathcal{A}_{\sigma\sigma'}(\vec p_1\vec p_2|\vec p_3\vec p_4) \equiv \mathcal{A}_{|i\rangle\to|f\rangle} = \gamma^{+}_{\vec p_1\sigma}\gamma^{+}_{\vec p_2\sigma'}\gamma_{\vec p_3\sigma'}\gamma_{\vec p_4\sigma}$$

use $|i) = e^{\hat S}|i)_0$ to express $\mathcal{A}$ as

$$\mathcal{A}_{i\to f} = {}_0\langle i|\hat T(E)|f)_0 \quad \text{with } \hat T = \hat V + \hat V \hat Q_f \hat G_0(E)\hat Q_i \hat V + \dots$$

many-body t-matrix

$$\left(G_0(E=E_i=E_f) = \frac{1}{E-\hat H_0}\right)$$

Special case $g_{\vec p\vec p'} \equiv \mathcal{A}_{\uparrow\downarrow}(\vec p,-\vec p|\vec p',-\vec p')$ pairing collisions

⚠ Resonant intermediate states.

D- Effective Hamiltonian

$$\hat{H}_{eff} = \sum_{\vec{p}\sigma} \varepsilon^{\circ}_{\vec{p}\sigma}\, \hat{n}_{\vec{p}\sigma} + \frac{1}{2V}\sum_{\vec{p}\sigma,\vec{p}'\sigma'} f_{\vec{p}\sigma,\vec{p}'\sigma'}\, \delta\hat{n}_{\vec{p}\sigma}\, \delta\hat{n}_{\vec{p}'\sigma'} + \sum_{\vec{p}\vec{p}'} g_{\vec{p}\vec{p}'}\, \hat{\gamma}^{+}_{\vec{p}\uparrow}\hat{\gamma}^{+}_{-\vec{p}\downarrow}\hat{\gamma}_{-\vec{p}'\downarrow}\hat{\gamma}_{\vec{p}'\uparrow}$$

$$+ \frac{1}{V}\sum_{\substack{(\vec{p}_1\vec{p}_2)\neq(\vec{p}_4\vec{p}_3)\\ \vec{p}_1\neq-\vec{p}_2}} \left(\mathcal{A}_{\uparrow\downarrow}(\vec{p}_1\vec{p}_2|\vec{p}_3\vec{p}_4)\, \hat{\gamma}^{+}_{\vec{p}_1\uparrow}\hat{\gamma}^{+}_{\vec{p}_2\downarrow}\hat{\gamma}_{\vec{p}_3\downarrow}\hat{\gamma}_{\vec{p}_4\uparrow} + \sum_{\sigma=\uparrow\downarrow}\frac{\mathcal{A}_{\sigma\sigma}}{2}(\vec{p}_1\vec{p}_2|\vec{p}_3\vec{p}_4)\, \hat{\gamma}^{+}_{p_1\sigma}\hat{\gamma}^{+}_{p_2\sigma}\hat{\gamma}_{p_3\sigma}\hat{\gamma}_{p_4\sigma} \right)$$

$$+\, \mathcal{O}(\hat{\gamma})^6$$

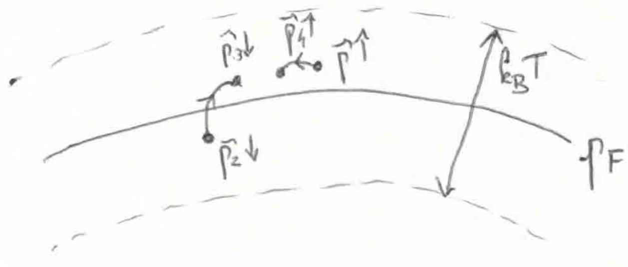
$\varepsilon^{\circ}_{\vec{p}\sigma}$, $f_{\vec{p}\sigma,\vec{p}'\sigma'}$ and $\mathcal{A}_{\sigma\sigma'}$ are the effective parameters that fully characterize the low-energy behavior.

Advantage of the qp picture: only on-shell couplings

$\rightarrow$ the qp gas ($\neq$ the p gas) is weakly-coupled

E- Quasiparticle lifetime

$$\begin{cases} n_{\vec{p}'\sigma'} = n^{eq}_{p} = \dfrac{1}{1+e^{(\varepsilon^{\circ}_{p'}-\mu)/k_BT}} \\ n_{\vec{p}\sigma} \neq n^{eq}_{p} \end{cases}$$



$$\Gamma_{\vec{p}\sigma} = \frac{2\pi}{V^2}\sum_{\substack{\vec{p}_2\vec{p}_3\\ \sigma'=\uparrow,\downarrow}} \delta\left(\varepsilon_{\vec{p}}+\varepsilon_{\vec{p}_2}-\varepsilon_{\vec{p}_3}-\varepsilon_{\vec{p}_4}\right) n^{eq}_{\vec{p}_2}\left(1-n^{eq}_{\vec{p}_3}\right)\left(1-n^{eq}_{\vec{p}_4}\right)\left|\mathcal{A}_{\sigma\sigma'}\left(\vec{p}\vec{p}_2|\vec{p}_3\vec{p}_4\right)\right|^2$$

$$= \frac{2\pi}{(2\pi)^6}\int (m^{*}p_F)^2\, d\varepsilon_2\, d\varepsilon_3\, n(\varepsilon_2)\left(1-n(\varepsilon_3)\right)\left(1-n(\varepsilon+\varepsilon_2-\varepsilon_3)\right)\int d\Omega_2\, d\Omega_3 \left|\mathcal{A}(\Omega_2,\Omega_3)\right|^2 \times \frac{m^{*}}{p_F^2}\,\delta\left(\cos\theta_2\left(\cos\frac{\theta_2}{2}-\cos\theta_3\right)\right)$$

$$\varepsilon_2 \rightarrow x_2 = \frac{\varepsilon_2-\mu}{k_BT}$$

$$\Gamma_{\vec{p}\sigma} = \underbrace{m^{*3}T^2\left\langle \mathcal{A}^2\right\rangle_{\Omega}}_{=\frac{1}{\tau_m},\ \text{mean collision time}} \times f\left(\frac{\varepsilon-\mu}{k_BT}\right) + \underbrace{(\text{deviation to FGR})}_{=\mathcal{O}(T^3)} + \underbrace{(3\text{-body collisions})}_{=\mathcal{O}(T^4)} + \dots$$

F_ Quasiparticle transport equation

* Prepare a weakly-excited state $\hat{\rho}(t) = \hat{\rho}_{eq} + \delta\hat{\rho}(t)$

Fermi-Dirac density matrix at temp T

* Study the phase-space distribution $n_{\vec{p}\sigma}(\vec{r},t) = \langle \gamma^+_{\vec{p}\sigma}(\vec{r})\, \gamma_{\vec{p}\sigma}(\vec{r}) \rangle_t$??

this semi-classical picture is possible only for $\lambda_{qp} < r_{typ}$

λ_{qp}: size of the qp; r_{typ}: spacial scale of the excitation

→ not true at low T

use $n_{\vec{p}\sigma}(\vec{q},t) = \langle \hat{\gamma}^+_{\vec{p}+\vec{q}\sigma}\, \gamma_{\vec{p}\sigma} \rangle_t$ instead

$$n_{\vec{p}\sigma}(\vec{q}) \xrightarrow[\text{Wigner transform}]{} n_{\vec{p}\sigma}(\vec{r})$$

* Equation of motion in Heisenberg picture

$$i\partial_t \hat{n}_{\vec{p}\sigma} = [\hat{n}_{\vec{p}\sigma}, \hat{H}_{eff}]$$ → not a closed system due to quartic terms

* Cumulant expansion

$$\hat{\gamma}^+_1 \hat{\gamma}^+_2 \hat{\gamma}_3 \hat{\gamma}_4 \simeq \underbrace{\langle \hat{\gamma}^+_1 \hat{\gamma}_4 \rangle_{eq}}_{\text{in } \hat{\rho}_{eq}} \hat{\gamma}^+_2 \hat{\gamma}_3 + \hat{\gamma}^+_1 \hat{\gamma}_4 \langle \hat{\gamma}^+_2 \hat{\gamma}_3 \rangle_{eq} - \langle \hat{\gamma}^+_1 \hat{\gamma}_3 \rangle_{eq} \hat{\gamma}^+_2 \hat{\gamma}_4 - \hat{\gamma}^+_1 \hat{\gamma}_3 \langle \hat{\gamma}^+_2 \hat{\gamma}_4 \rangle_{eq}$$

$$+ (\hat{\gamma}^+_1 \hat{\gamma}^+_2 \gamma_3 \gamma_4)_c + (\text{cst. terms})$$

* Linearized transport eq

$$\underbrace{i\partial_t n_{\vec{p}\sigma} + \overbrace{(\overset{\circ}{\varepsilon}_{\vec{p}+\vec{q}} - \overset{\circ}{\varepsilon}_{\vec{p}})}^{H^{(2)}_{eff}} n_{\vec{p}\sigma} - (n^{eq}_{\vec{p}+\vec{q}} - n^{eq}_{\vec{p}}) \Big(\overbrace{\frac{1}{V}\sum_{\vec{p}'\sigma'} f_{\vec{p}\sigma,\vec{p}'\sigma'}\, n_{\vec{p}'\sigma'}}^{H^{(4)}_{eff}\ \text{contract}} + \overbrace{U^{ext}_\sigma(t)}^{\text{external drive}} \Big)}_{\approx \omega \approx v_F q \ll \varepsilon_F} = \underbrace{\overbrace{i\, I_{coll}[\{n_{\vec{p}'\sigma'}\}]}^{\text{Markovian treatment of } (\hat{a}\hat{b}\hat{c}\hat{d})}}_{\approx \frac{1}{\tau_m} \ll \varepsilon_F}$$

(low-energy / long wavelength "slowly varying regime")

In frequency domain $\left(i\partial_t n \to \omega n\right)$, this becomes a linear integral equation $\left(n_{\vec{p}\sigma} \text{ is the unknown function}\right)$

→ Solve it for response functions

$$\rho(\vec{q},\omega) \equiv \sum_{\vec{p}\sigma} n_{\vec{p}\sigma} = \chi_\rho(\vec{q},\omega)\left(U_\uparrow + U_\downarrow\right)$$
$$\rho(\vec{q},\omega) \equiv \sum_{\vec{p}} \left(n_{\vec{p}\uparrow} - n_{\vec{p}\downarrow}\right) = \chi_\rho(\vec{q},\omega)\left(U_\uparrow - U_\downarrow\right)$$

* Collectives modes: poles of $\omega \longmapsto \chi(\vec{q},\omega)$

Collisionless regime $\boxed{\omega\tau_m \gg 1}$

$f_{\uparrow\uparrow} + f_{\uparrow\downarrow} < 0$ for $a < 0$

→ no zero sound in χ_ρ

$f_{\uparrow\uparrow} - f_{\uparrow\downarrow} > 0$

→ spin zero mode in χ_ρ

but $\frac{c}{v_F} - 1 \approx e^{-\pi/k_F|a|}$

→ Very close to the particle-hole continuum and very low spectral weigth

→ difficult to observe

Hydrodynamic regime $\boxed{\omega\tau_m \ll 1}$

$n_{\vec{p}\sigma} \approx n_{\vec{p}\sigma}^{\text{cons}}$ with $I\left[n_{\vec{p}\sigma}^{\text{cons}}\right] = 0$

Transport eq → Fermi Liquid version of the Navier-Stokes equations

- First sound resonance in χ_ρ
- Dissipation via transport coefficients
  - η shear viscosity
  - D spin diffusivity
  - K Thermal conductivity
- Measurements (in part. via sound attenuation) in MIT, Yale, North Carolina

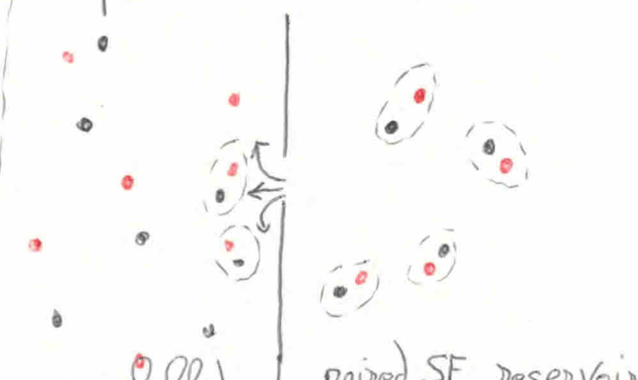
G- Pairing instability

Imagine that $\hat{\delta\rho}(t)$ is such that

$$d_{\vec p}(\vec q,t) \equiv \left\langle \hat\gamma_{-\vec p-\vec q\downarrow}\,\hat\gamma_{\vec p\uparrow}\right\rangle_t \neq 0$$

Wigner transform of $\left\langle \psi_\downarrow\left(\vec r+\frac{\vec s}{2}\right)\psi_\uparrow\left(\vec r-\frac{\vec s}{2}\right)\right\rangle$

Experimental situation



Second order cumulant eq of motion for $d_{\vec p}$

$$i\partial_t d_{\vec p} = \left(\epsilon^0_{\vec p}+\epsilon^0_{\vec p+\vec q}-2\mu\right)d_{\vec p} + \left(1-n^{eq}_{\vec p}-n^{eq}_{\vec p+\vec q}\right)\left(\sum_{\vec p'} g_{\vec p\vec p'}\,d_{\vec p'}+U_{pair}\right)$$

Pairing susceptibility $\quad \delta\Delta_{\vec p} = \sum_{\vec p'} g_{\vec p\vec p'}\,d_{\vec p'} \equiv \chi_\Delta(\vec p,\vec q,\omega)\,U_{pair}$

Thouless criterion $\quad \chi_\Delta\left(\cancel{\vec p},\vec q=0,\omega=0\right)=0 \Leftrightarrow$ 2nd order phase transition

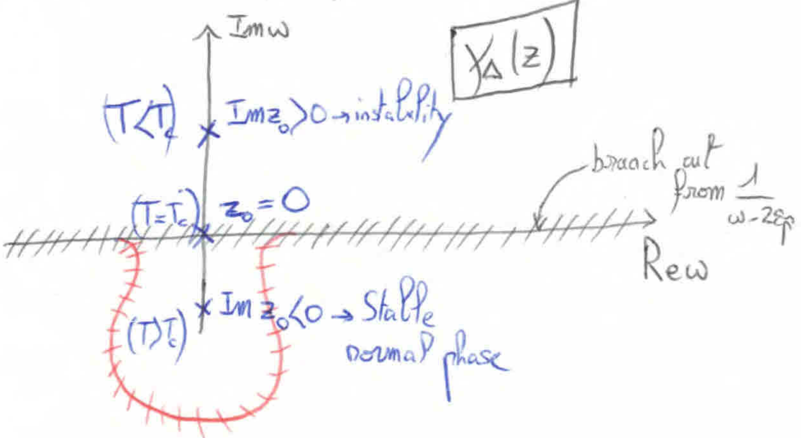
for q=0

$$\chi_\Delta^{-1}(\vec q=0,\omega)=1-\bar g_{pair}\sum_{\vec p}\left(\frac{1-2n^{eq}_{\vec p}}{\omega-2\epsilon_{\vec p}}+\frac{m}{p^2}\right)$$

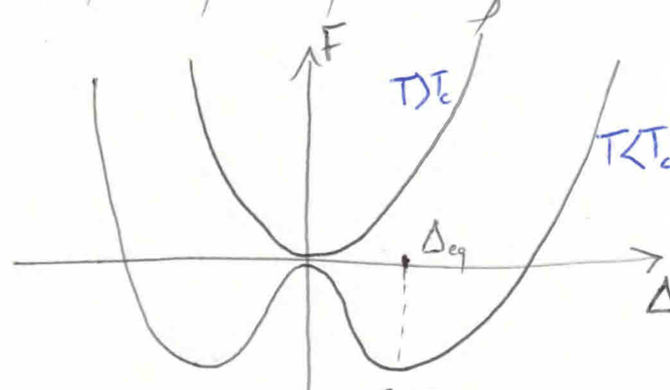
perturbatively $\bar g_{pair}=g+\alpha g^2+\dots$

$$\chi_\Delta^{-1}(\omega=0,T=T_c)=0 \Leftrightarrow \log\frac{T_c}{T_F}=-\frac{\pi}{2|a|}+C_{GMB}+O\left(\frac{1}{a}\right)$$

. Pole in the complex plane



Profile of the free energy



$$\left.\frac{\partial^2 F}{\partial\Delta^2}\right|_{\Delta=0} \propto \chi_\Delta^{-1}(\omega=0)$$