Problem 7

WARP-drive

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Background "WARP 5, Mr. Sulu".
What a dream. Travelling as fast as one wishes. But nature sets a boundary, the speed of light. Eventhough there might seem no chance for a 'WARP-drive', in the 90ies M. Alcubierre published a geometry of an Einsteinian space-time that would allow for transport of material objects over far spatial distances in arbitrary short amount of time. [1]
In this problem, we will study its metric and discuss the need for a certain arrangement of 'exotic matter' for such a space-time. We will work in two spatial dimensions. That eases the problem without loss of essential features of Alcubierre's ideas. In this problem, the signature convention on the metric is $(-,+,+)$ and $\mathrm{d}x^0 = \mathrm{d}t$, $\mathrm{d}x^1 = \mathrm{d}x$, and $\mathrm{d}x^2 = \mathrm{d}y$.

Let's begin by discussing the Alcubierre metric. It is constructed as follows

$$\mathrm{d}s^2 = \sum_{\mu=0}^{2}\sum_{\nu=0}^{2} g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu,$$
$$\{g_{00} = -(1-\beta^2(t,x,y)), \quad g_{01} = g_{10} = \beta(t,x,y), \quad g_{11} = g_{22} = 1, \quad g_{\mu\nu} = 0 \quad \text{else}\}. \tag{7.1}$$

$\beta(t,x,y)$ models the 'WARP-bubble' in the $(x,y)-$plane for a given time.
$\beta(t,x,y) = 0$ aside the bubble.
If you want to see an explicit expression of how β may look like, confer the very end of this problem, however, you don't need that information for this problem!

PART I

This metric may appear weird at first sight. It has 'side elements' and g_{00} seems to change sign, depending on where one evaluates the metric in the parameterisation. So, let's check if it is a proper metric.

a) [*0.5 points*] Show that g is asymptotically flat, i. e. it converges to the Minkowski metric far aside the 'WARP-bubble' for any given time.
Hint: *The Minkowski metric reads* $\{\eta_{00} = -1, \quad \eta_{11} = \eta_{22} = 1, \quad \eta_{\mu\nu} = 0 \quad \text{else}\}$.

b) [*1 point*] Proof that its determinant reads

$$g = \det(g_{\mu\nu}) = -1, \tag{7.2}$$

and find the inverse metric $g^{\mu\nu}$. Note $\sum_{\alpha=0}^{2} g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu_\nu$, where $\delta^\mu_\nu = 1$ for $\mu = \nu$, and 0 otherwise.
Hint: *you can use the fact, that g 'looks like' a block diagonal matrix.*

Finally, the metric has to support time-like vector fields. A vector field $\{u^\mu\}$ is time-like iff

$$\sum_{\mu=0}^{2}\sum_{\nu=0}^{2} g_{\mu\nu}u^\mu u^\nu < 0. \tag{7.3}$$

c) [*1 point*] Check that

$$\{u^0 = 1, \quad u^1 = -\beta, \quad u^2 = 0\}, \tag{7.4}$$

is indeed such a time-like vector field. A trajectory to which u is tangential obeys the principle of special relativity to not exceed the speed of light locally.

In that context, describe **Figure 7.1**.

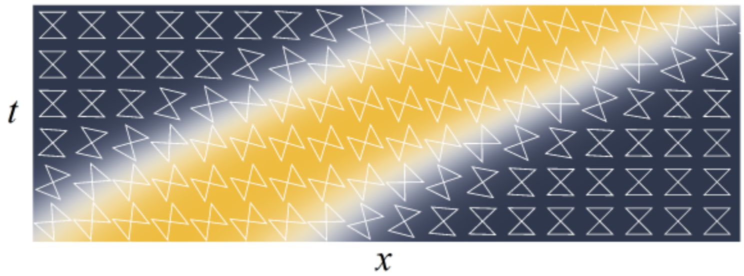

Figure 7.1 Cross-section of Alcubierre space-time at $y = 0$ for far away observer. The cones are the local light cones. [5]

PART II

$\Gamma_{\nu\alpha\beta}$ are the Christoffel symbols (of first kind), which convey the gravitational field.

$$\Gamma_{\nu\alpha\beta} := \tfrac{1}{2}\left(-\partial_\nu g_{\alpha\beta} + \partial_\alpha g_{\beta\nu} + \partial_\beta g_{\nu\alpha}\right) \tag{7.5}$$

[2]. Einstein supposed that the energy momentum density tensor $T^{\mu\nu}$ sources a non-trivial curvature to the geometry of our space-time. The component T^{00} in $T^{\mu\nu}$ equals the energy density.
Note that matter, as we know it, is represented by an energy density field that is strictly positive!
Due to energy momentum conservation and symmetry constraints the simplest possible way to formulate such a relation between the energy momentum density tensor and curvature is

$$\kappa \cdot T^{\rho\sigma} = R^{\rho\sigma} - \tfrac{1}{2} g^{\rho\sigma} R, \tag{7.6}$$

where κ is a positive constant. $R = \sum\limits_{\tau=0}^{2}\sum\limits_{\zeta=0}^{2} g_{\tau\zeta} R^{\tau\zeta}$ is the Ricci scalar, and $R^{\tau\zeta}$ is the Ricci tensor. We are going to use Einstein's sum convention, i. e. if an upper index and a lower index have the same letter and occur in a product of two tensors, it is automatically summed over: $R = g_{\tau\zeta} R^{\tau\zeta}$. The Ricci tensor can be constructed from the Riemann curvature tensor $R_{\alpha\beta\mu\nu}$, via

$$R^{\rho\sigma} = g^{\rho\beta} g^{\sigma\nu} R_{\beta\nu}, \qquad R_{\beta\nu} = g^{\alpha\mu} R_{\alpha\beta\mu\nu}. \tag{7.7}$$

The Riemann curvature tensor contains all curvature information of our space-time and can be written as

$$\begin{aligned} R_{\alpha\beta\mu\nu} := \tfrac{1}{2}\left(\partial_\alpha\partial_\nu g_{\mu\beta} - \partial_\alpha\partial_\mu g_{\beta\nu} - \partial_\beta\partial_\nu g_{\mu\alpha} + \partial_\beta\partial_\mu g_{\alpha\nu}\right) + \\ + g^{\sigma\rho}\left(\Gamma_{\rho\alpha\nu}\Gamma_{\sigma\mu\beta} - \Gamma_{\rho\beta\nu}\Gamma_{\sigma\mu\alpha}\right). \end{aligned} \tag{7.8}$$

Note exactly the order of indices! cf. [2].
Admittedly, this expression is overwhelming, thus this part of the problem is mostly about to find

an elegant way to not evaluating the most of these components and terms.

d) [*0.5 points*] Start out by showing

$$\Gamma_{\alpha\mu\nu} = \Gamma_{\alpha\nu\mu}, \tag{7.9}$$

$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta} = -R_{\mu\nu\beta\alpha}. \tag{7.10}$$

with using $g_{\mu\nu} = g_{\nu\mu}$ and Eqn(7.5). Hint: *Symmetry relations of tensors transform to all coordinate frames. Only for this task you may go to a 'Riemann frame', in which the Christoffel symbols vanish!*

e) [*4 points*] Calculate all non-trivial Christoffel symbols.

The trivials are $0 = \Gamma_{022} = \Gamma_{101} = \Gamma_{111} = \Gamma_{112} = \Gamma_{122} = \Gamma_{202} = \Gamma_{211} = \Gamma_{212} = \Gamma_{222}$, and those that can be obtained via the symmetry relation.
You do not have to validate these! As an intermediate result: $\Gamma_{012} = \frac{1}{2}\partial_y \beta$.

f) [*0.5 points*] Only from these (anti-)symmetries you can work out what components of the Riemann tensor $R_{\alpha\beta\mu\nu}$ are non-vanishing in general. Indicate them with a '$\times$' in the following table :

$\alpha\beta$ \ $\mu\nu$	00	01	02	11	12	22
00						
01						
02						
11						
12						
22						

For the remainder of the problem we want to work out what actually is the arrangement of matter we need, to make our space-time to become an Alcubierre space-time. To that end let's calculate T^{00}, and we will use some short cutting tricks, to not waste time on calculating Riemann components which we do not need.

g) [*1 point*] Write down R^{00} in terms of the explicit expression of the inverse metric and keep the components of $R_{\beta\nu}$ as they come, like:

$$R^{00} = g^{0\beta} g^{0\nu} R_{\beta\nu} = g^{00} g^{00} R_{00} + ... = (-1)(-1) R_{00} + ... \tag{7.11}$$

Proceed in the same way for the Ricci scalar R.

h) [*1.5 points*] Now, calculate the energy density $\kappa T^{00} = R^{00} - \frac{1}{2} g^{00} R$ by expanding all occurring Ricci-tensor components in terms of Riemann-tensor components, and show that this energy density is non-positive! **Figure 7.2** is an image of the resulting energy density.
Hint: *Do not calculate the Riemann components until the very end. You should find* $\kappa T^{00} = R_{2121}$.

A comment on how $\beta(t, x, y)$ might look like.

$$\beta(t, x, y) = -v_0 \cdot f(t, x, y), \tag{7.12}$$

where $v_0 = const$. f is a smoothed Heaviside function on a circular domain (of radius r_0) in the $(x, y)-$plane, and has a value of 1 inside the domain and 0 outside, with a smooth but narrow transition zone (of width w) between these two values, so, f is a continuously differentiable function to at least second order. Its center is shifted along the x-axis over time.

$$f(t, x, y) = \Theta_w \left(\frac{r_0 - \sqrt{(x - v_0 \cdot t))^2 + y^2}}{w} \right). \tag{7.13}$$

x, y and t is the choice of a global parameterisation of Alcubierre's space-time, used by an observer that is spatially far aside the 'WARP-bubble' for any given time.

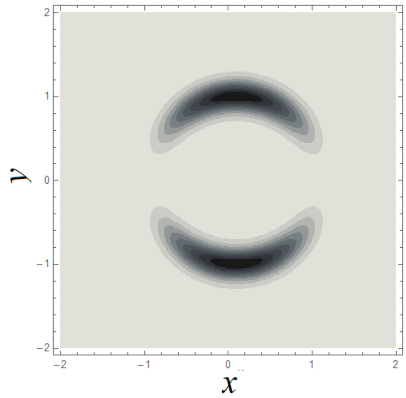

Figure 7.2 Arrangement of 'exotic matter' in (x, y)-plane at $t = 0$ for $w = 0.25$. The darker the gray tone, the higher the density of 'exotic matter'. In 3D, a space ship being placed in the WARP-bubble given by f, would be surrounded by a 'donut' of 'exotic matter' with the symmetry axis being *along the x-axis*! [5]