Problem 3

Anti-reflection coating for a disordered medium

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Background When light impinges on a disordered medium – say, a sugar cube – it is scattered in a seemingly random manner. In principle, the coherent scattering is determined by the microscopic structure of the disordered medium; yet, even for everyday objects as the said sugar cube, the structure is far too complex to be characterized on such microscopic scales. Recently, it was shown, however, that it is possible to bypass the complex scattering : By simply measuring the reflection properties of the disordered medium, it is possible to determine conditions for an anti-reflection coating that renders the complex medium perfectly transmitting. Here, we will explore the underlying concept of this coating in a simplified one-dimensional model.

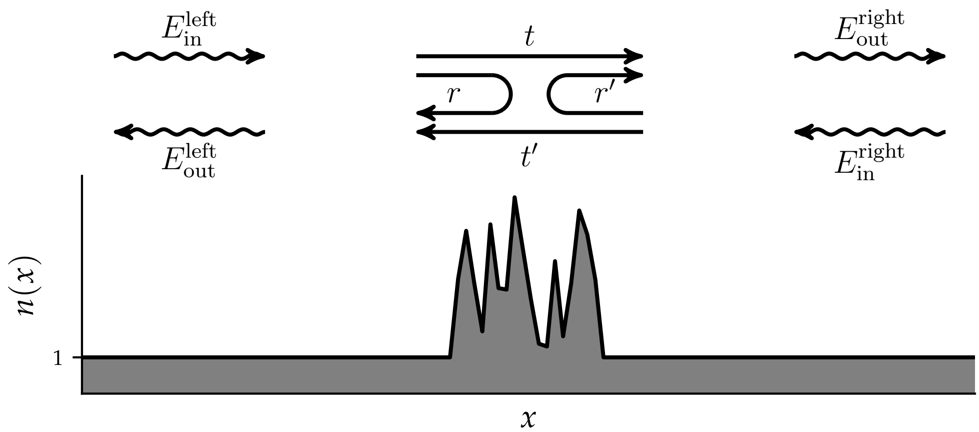

Figure 3.1 Scattering light at a disordered barrier.

A one-dimensional disordered medium can be modeled as a refractive index barrier varying randomly in space (see Fig. 3.1). We consider the case where the barrier is illuminated with a single plane wave of some frequency. The plane wave may impinge on the barrier from both sides, and may be reflected or transmitted in either case. Therefore, all relevant information outside the barrier can be captured by a so-called scattering matrix $\boldsymbol{S} \in \mathbb{C}^{2\times 2}$ that connects the in- and outgoing plane wave amplitudes,

$$\begin{pmatrix} E_{\text{out}}^{\text{left}} \\ E_{\text{out}}^{\text{right}} \end{pmatrix} = \boldsymbol{S} \begin{pmatrix} E_{\text{in}}^{\text{left}} \\ E_{\text{in}}^{\text{right}} \end{pmatrix} . \tag{3.1}$$

The entries of the scattering matrix can be linked to different scattering events that are illustrated in the sketch,

$$\boldsymbol{S} = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} . \tag{3.2}$$

a) [*2 points*] Before we consider the anti-reflection coating, we want to study some elementary properties of the scattering matrix. We assume the scattering matrix to be unitary, $\boldsymbol{S}^\dagger \boldsymbol{S} = \mathbf{1}$. Show that the following two relations hold for the reflection and transmission coefficients

$$tt^* + r'r'^* = 1, \tag{3.3}$$
$$rt^* + t'r'^* = 0. \tag{3.4}$$

Likewise, the unitarity implies that $|r|^2 + |t|^2 = |r'|^2 + |t'|^2 = 1$. Which physical property of the scattering process does this correspond to?

b) [*2 points*] Now we place the supposed anti-reflection coating, which is just another (for now arbitrary) dielectric material, in front of our disordered barrier (see Fig. 3.2). The coating as well as the barrier are both described by a unitary scattering matrix, i.e., the relations derived in (a) hold for the coating scattering matrix $\boldsymbol{S}_\mathrm{c}$ and the barrier scattering matrix $\boldsymbol{S}_\mathrm{b}$,

$$\boldsymbol{S}_\mathrm{c} = \begin{pmatrix} r_\mathrm{c} & t'_\mathrm{c} \\ t_\mathrm{c} & r'_\mathrm{c} \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{S}_\mathrm{b} = \begin{pmatrix} r_\mathrm{b} & t'_\mathrm{b} \\ t_\mathrm{b} & r'_\mathrm{b} \end{pmatrix}, \tag{3.5}$$

respectively. Given the scattering matrix of the coating and the scattering matrix of the barrier, show that the total reflection r_tot for light impinging on the joint structure from the left hand side (see Fig. 3.2) is given by

$$r_\mathrm{tot} = r_\mathrm{c} + \frac{t_\mathrm{c} r_\mathrm{b} t'_\mathrm{c}}{1 - r'_\mathrm{c} r_\mathrm{b}}. \tag{3.6}$$

Note that coating and disordered barrier are close enough together such that the phase acquired upon propagating in the vacuum between them can be neglected.
Hint: Make sure to take into account all multiple scattering contributions.

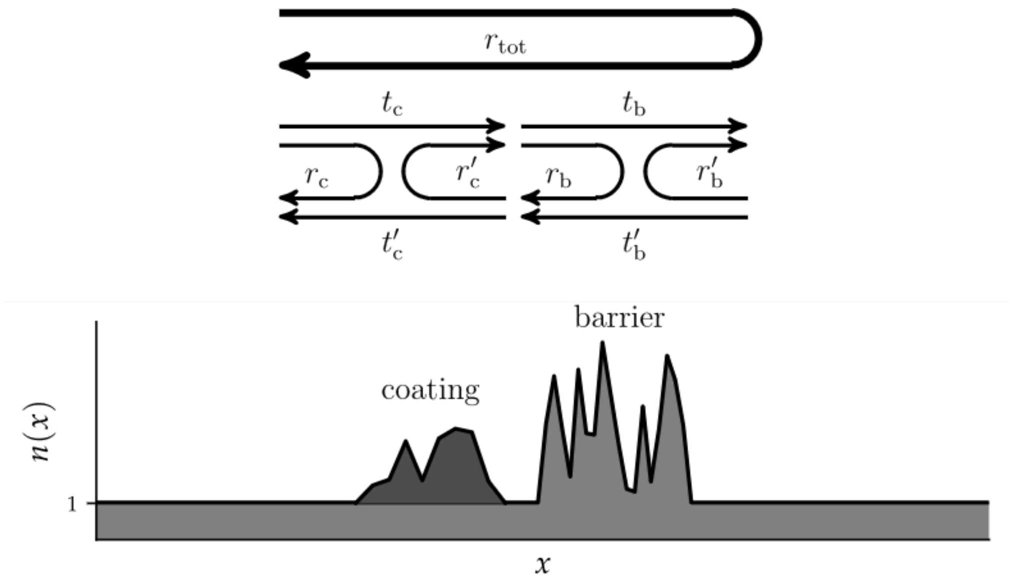

Figure 3.2 Disordered barrier (right) with anti-reflection coating (left).

c) [*1 point*] Using the formulae derived in (a) and (b), show that under the condition that $r'^*_\mathrm{c} = r_\mathrm{b}$, the total reflection is completely suppressed, i.e., $r_\mathrm{tot} = 0$. If this condition is fulfilled, the impinging monochromatic wave is perfectly transmitted through the joint system.

d) [*3 points*] The ultimate objective is to design an anti-reflection coating that suppresses the total reflection for a given disordered barrier. For simplicity, we use a rectangular dielectric slab of height n_0 and width d as the anti-reflection coating. We consider the limit where $d \to 0$ and $n_0 \to \infty$, such that $\eta \propto n_0 d^2, \eta \in (-\infty, \infty)$ remains constant. The value η then parametrizes the reflectivity of the idealized dielectric slab. It can be shown that the associated scattering matrix of the slab is given by

$$\boldsymbol{S}_\mathrm{c} = \begin{pmatrix} \dfrac{i\eta}{2-i\eta} & \dfrac{2}{2-i\eta} \\ \dfrac{2}{2-i\eta} & \dfrac{i\eta}{2-i\eta} \end{pmatrix}. \tag{3.7}$$

First, verify the unitarity of this scattering matrix. The reflectivity r_b of an arbitrary disordered barrier may take values within the unit circle in the complex plane. Find all values of r_b for which the idealized dielectric slab can serve as an anti-reflection coating. Show that these values lie on a circle with center at $-\frac{1}{2}$, and mark this circle in the complex plane.

e) [*2 points*] Suppose we cannot find a suitable coating for a given disordered barrier. As it turns out, there are still some input configurations $(E_\mathrm{in}^\mathrm{left}, E_\mathrm{in}^\mathrm{right})$ that remain – up to a phase – unchanged by the scattering at the disordered barrier, i.e., the barrier acts as a phase shifter, $(E_\mathrm{in}^\mathrm{left}, E_\mathrm{in}^\mathrm{right}) \to e^{i\phi}(E_\mathrm{in}^\mathrm{right}, E_\mathrm{in}^\mathrm{left})$. These input states can be interpreted as *scattering invariant modes* [4].

We want to exemplify this by taking the idealized dielectric slab instead of a disordered barrier. For the scattering matrix (3.7), find all (linearly independent) scattering invariant modes and specify which phases they acquire upon scattering at the idealized dielectric slab.