DOPPLERS

Problem 1

White Dwarves and the Chandrasekhar mass

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Background White Dwarves are remnants of burnt out stars, which are not any more stabilised against their gravitational attraction by thermal pressure but by Fermi or degeneracy pressure of the electrons. For relativistic electrons there is an upper bound for the mass of such stars, which was found by 19-years-old Subrahmanyan Chandrasekhar (1910-1995). This problem shows how the Chandrasekhar mass boundary can be determined in a simple way.

Degenerate electrons The Pauli exclusion principle states that each cell in phase space can be occupied by at most two electrons.

a) [*1 point*] *Determine the phase space volume accessible to an ideal gas confined to a fixed volume V with a momentum cutoff p_F.* The phase space is divided into cells with an edge length of the Planck constant h. *Show that the number of cells accessible to the gas is given by*

$$\Omega = \frac{4\pi}{3h^3} V p_{\mathrm{F}}^3 \tag{1.1}$$

b) [*1 point*] Due to the Pauli exclusion principle each cell can at most be occupied by two electrons. *Determine the relation between the Fermi momentum p_F and the maximal number of electrons N_e in a volume V. Using the result determine two expressions for the Fermi energy $E_F = E(N_e)$, one for a non-relativistic electron gas and one for an ultra-relativistic electron gas.*

c) [*2 points*] *Determine the average energy of the electrons, which corresponds to the internal energy U, and with this the pressure*

$$P = -\frac{\partial U}{\partial V} \ . \tag{1.2}$$

In particular, *show that in both cases the pressure is related to the density through a polytropic equation of state*

$$P = P_0 \left(\frac{\rho}{\rho_0} \right)^{\alpha} \tag{1.3}$$

and determine the plytropic index α in both cases.

Lane-Emden equation In hydrostatic equilibrium the outward pressure compensates the gravitational force. In that case the hydrostatic equation

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM(r)}{r^2} \tag{1.4}$$

holds, where pressure P and density ρ are functions of the radius r and $M(r)$ is the mass enclosed in radius r.

d) [*1 point*] Express the mass $M(r)$ in terms of the density ρ and *derive the relation*

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -4\pi G\rho \tag{1.5}$$

using eq. (1.4).

e) [*2 points*] Assume now a polytropic equation of state of the form given in eq. (1.3). Introduce the polytropic index

$$n = \frac{1}{\alpha - 1} \tag{1.6}$$

and replace the density ρ using the definition

$$\frac{\rho}{\rho_0} \equiv \theta^n \ . \tag{1.7}$$

Using these definitions *bring eq.* (1.5) *into the form of the dimensionless Lane-Emden equation*

$$\frac{1}{x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\frac{\mathrm{d}\theta}{\mathrm{d}x}\right) + \theta^n = 0 \ , \tag{1.8}$$

where the length scale

$$r_0 \equiv \sqrt{\frac{(n+1)P_0}{4\pi G\rho_0^2}} \tag{1.9}$$

was used to introduce the dimensionless radius $x \equiv r/r_0$. For our purposes the length scale r_0 is particularly important.

f) [*3 points*] For most values of the polytropic index n the Lane-Emden equation has to be solved numerically. Fig. 1 shows the solution for $\alpha = 4/3$, i.e. $n = 3$, with boundary conditions $\theta = 1$, $\theta' = 0$ at $x = 0$. Moreover, $x^2\theta'(x)$ is shown, which could be useful later.

The figure shows the density profile $\theta(x)$ for a star, which is stabilised by an ultra-relativistic degenerate electron gas. *Estimate the mass of such a White Dwarf using the solution of the Lane-Emden equation in Fig. 1. In your result, substitute the Planck-Mass*

$$m_{Pl} = \sqrt{\frac{\hbar c}{G}} \tag{1.10}$$

and bring the result to the simplest possible form. Express the result in units of the solar mass ($M_\odot = 2 \cdot 10^{33}$ g).

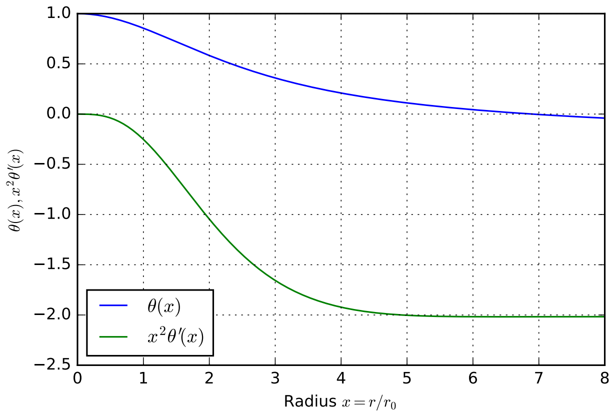

Figure 1 Numerical solution of the Lane-Emden equation (1.8).